Playing Schmidt Games with Markov Partitions
A Study of Nondense Orbits

Jim Tseng
Brandeis University

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Background

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- $\sigma$ - The (probability) volume measure on $M$.

$T : M \to M$ - A $C^2$-expanding map, which means (perhaps after a suitable smooth change of Riemannian metric), that there exists $\lambda > 1$ such that $\|D_x T(v)\| \geq \lambda \|v\|$ for all $x \in M$ and for all $v \in T_x M$.

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The Basics of Nondense Orbits

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- By the Birkhoff Ergodic Theorem, $\sigma(A) = 0$. 
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Our main object of study:

\[ A := \{ x \in M \mid \overline{\mathcal{O}_T^+(x)} \not\subseteq M \}. \]

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However ...
The Hausdorff Dimension (HD) of Nondense Orbits

**Theorem (M. Urbański, 1991)**

\[ HD(A) = \dim(M). \]
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Another interesting theorem:

- \( f \) - a piecewise expanding map of an interval \( I \).
- \( B \) - a subset of \( I \).

**Theorem (D. Dolgopyat, 1997)**

If \( HD(B) < 1 \), then the set of points whose forward orbits eventually avoid \( B \) has Hausdorff dimension 1.
The First Main Tool: Markov Partitions

A set \( \{R_1, \cdots, R_s\} \) of subsets of \( M \) is a Markov partition for \( T \) (a \( C^2 \)-expanding self-map) if:

- They cover \( M \).
- Each element is the closure of its interior.
- Interiors of the elements are pairwise disjoint.
- The boundary of each element has zero volume.
- The diameters of all elements are small enough so that \( T \) is injective on all elements.
- \( T(R_i) \) is a union of elements.
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- $\tilde{\Sigma}$ - the set of all one-sided infinite sequences $\alpha := \alpha_0 \alpha_1 \cdots$ in the alphabet $\{1, \cdots, s\}$. 

- $\phi$ - the left shift operator.
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The Markov partition provides a semi-conjugacy

$$\pi : (\Sigma, \varphi) \rightarrow (M, T) \quad \alpha \mapsto R_\alpha$$

where

$$R_\alpha := R_{\alpha_0} \cap T^{-1}(R_{\alpha_1}) \cap \cdots \cap T^{-n}(R_{\alpha_n}) \cap \cdots .$$
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Note that $R_\alpha$ is a point in $M$. 
The Simplest Example of a Markov Partition

Let

\[ S^1 = \mathbb{R}/\mathbb{Z} \]

denote the circle and

\[ \{ R_0 := [0, \frac{1}{2}], R_1 := [\frac{1}{2}, 1] \} \]

is a Markov partition for \( E_2 \).

Continuing, one creates infinite sequences \( \alpha \) that correspond to a binary expansion of the real number \( R_\alpha \).
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The Second Main Tool: Schmidt Games

- Introduced by W. Schmidt in 1966.

Let $0 < \alpha < 1$ and $0 < \beta < 1$. Let $S$ be a subset of a complete metric space $M$. Two players alternate choosing nested closed balls $B_1 \supset W_1 \supset B_2 \supset W_2 \cdots$ on $M$. Require $\text{radius}(W_n) = \alpha \cdot \text{radius}(B_n)$ and $\text{radius}(B_n) = \beta \cdot \text{radius}(W_{n-1})$. The second player wins if the intersection of these balls lies in $S$. The set $S$ is called $\alpha$-winning if the second player can always win for any $\beta$. 

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- Countable intersections of $\alpha$-winning sets are again $\alpha$-winning.
An Example of a Winning Set

- Let $T^n = \mathbb{R}^n / \mathbb{Z}^n$ be the $n$-torus.
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- Let $\rho$ be a semisimple surjective endomorphism of $\mathbb{T}^n$. 

**Theorem (S. G. Dani, 1988)**

The set of points whose forward orbit closures miss the identity element in $\mathbb{T}^n$ is $\frac{1}{2}$-winning.

**Corollary (S. G. Dani, 1988)**

The set of points whose forward orbit closures under any semisimple surjective endomorphism that miss $\mathbb{Q}^n$ is $\frac{1}{2}$-winning.
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A corollary:

Let $Q = \mathbb{Q}^n / \mathbb{Z}^n$. 
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The Idea of the Proof of $HD(A) = \dim(M)$

An outline of Urbański’s elegant proof:

$$\Sigma(n) := \{\gamma_0 \cdots \gamma_n | \cap \bigg( \bigcap_{j=1}^n \text{Int} R_{\gamma_j} \bigg) \neq \emptyset \ \forall \ 1 \leq j \leq n \}.$$ 

Let $\gamma \in \Sigma(n)$ and consider $U := \text{Int} R_{\gamma}$. Let $\alpha \in \Sigma$. The point $R_{\alpha}$ visits $U$ if and only if $\gamma$ appears as a substring of $\alpha$.

Take all infinite sequences $\Sigma_\gamma := \{\alpha \in \Sigma | \gamma \text{ does not appear as a substring of } \alpha \}$. Adapting a version of a lemma by C. McMullen, Urbański tries to show that the set $\bigcup_{n=0}^\infty \bigcup_{\gamma \in \Sigma(n)} \{R_{\alpha} | \alpha \in \Sigma_\gamma \}$ has full Hausdorff dimension. There is, however, a problem: He does not correctly construct $\Sigma_\gamma$. 

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- Therefore the proof fails.
My correction is simply to show that $\gamma$ cannot appear anywhere as a substring in $\alpha$. 
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Using the correction, Urbański’s original theorem can be slightly strengthened:

**Theorem**

Choose $x_0$ from a certain set of full measure in $M$. Then

$$F_T(x_0) := \{ x \in M \mid x_0 \notin \mathcal{O}^+_T(x) \}$$

has full Hausdorff dimension.
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- Using the correction, Urbański’s original theorem can be slightly strengthened:

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- However, as the following technical lemma shows, making the correction can be tricky.....
The Technical Lemma: An Illustration

**Lemma (T.)**

Let $N \geq n \geq 8s - 4$. Let $\gamma$ be any $n$-string such that $\gamma_{n-1}$ is nondegenerate except those of the following kind:

$$\gamma = a^0 \cdots a^m$$

where

$$a^0 = \ldots = a^{m-1}$$

are general blocks and either

$a^m$ is a general block not equivalent to $a^0 a^0$

or

$a^m$ is a double general block not equivalent to $a^0 a^0$.

And let $\alpha$ be a $N$-string such that no match of $\gamma$ with $\alpha$ exists. Then there exists a choice of substrings $b^0$ and $b^1$ of length at most $s$ such that, for any letters $\beta_0, \beta_1, \ldots, \beta_k$, no match of $\gamma$ with the $N+n$-string $\alpha b^0 b^1 \beta_0 \cdots \beta_k$ exists.
Note that Urbański has come up with a second, shorter correction of his original theorem.
A Second Correction

- Note that Urbański has come up with a second, shorter correction of his original theorem.
- But, my original correction leads to ....
**The Main Result: A Generalization in 1-D**

**Theorem:**

Let \( x_0 \in S^1 \). Then

\[
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The Main Result: A Generalization in 1-D

**Theorem (T.)**

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These generalize Urbański’s theorem and, in part, Dani’s theorem in dimension one.
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- Using the two forms of partial control, we carefully fit the two nested sequences together.□
Supplemental
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  - Example. $B \in \mathcal{B}_M$ such that $\mu(B) < \infty$ and $T(B) \subset B$.
- A measure-preserving dynamical system is **ergodic** (for $\mu$) if every essentially $T$-invariant subset is null or conull.
Theorem (BET)

If $f : X \to X$ is an ergodic $\mu$-preserving map, $\mu(X) = 1$, and $\varphi \in L_1(X)$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_X \varphi \, d\mu$$

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- If \( \varphi \) is the characteristic function of an open set \( U \) in \( M \), BET implies the points that miss \( U \) have zero measure.
- Our \( M \) is second countable, thus \( \sigma(A) = 0 \).
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Let \( m_d^\alpha(A) \) be the greatest lower bound of all sums

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J. Tseng (Brandeis University) Schmidt Games and Markov Partitions Penn State 19 October 2007 23 / 23