Math 111a, Fall 2008, Homework # 3

Banach Spaces

1. (Lang, Problem 14(d) on p. 47) Let A be a subset of a Banach space E, let $x \in A$ and $y \notin A$. Show that there exists $t \in [0,1]$ such that $tx + (1-t)y \in \partial A$.

2. Let $\alpha = (a_n)$ be such that the sequence (na_n) belongs to l^p for some $p < \infty$. Prove that $\alpha \in l^1$.

3. Let $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$. Prove that for any three sequences $\alpha = (a_n), \beta = (b_n), \gamma = (c_n)$ one has $\|\alpha\beta\gamma\|_1 \leq \|\alpha\|_p\|\beta\|_q\|\gamma\|_r$, where one defines $\alpha\beta\gamma \stackrel{\text{def}}{=} (a_n b_n c_n)$.

4. (a) (Lang, Problem 6(a) on p. 91) Prove that any finite-dimensional vector subspace of a normed vector space is closed.

(b) (Lang, Problem 12 on p. 92) Let E be an infinite-dimensional Banach space, and let $\{x_n\}$ be a sequence of linear independent unit vectors. Show that there exists an element in the closure of the space generated by all x_n which does not lie in any subspace generated by a finite number of x_n. (Look in Lang's book for a hint.)

(c) By combining (a) and (b), prove that a Banach space is finite-dimensional if and only if all its subspaces are closed.

5. Let $C^1[0,1]$ stand for the space of continuously differentiable functions on $[0,1]$ (that is, $C^1[0,1] \stackrel{\text{def}}{=} \{f \mid f' \text{ exists and is continuous on } [0,1]\}$). Consider the following four "norms" on this space:

$$\|f\| = \sup_{[0,1]} |f(x)| + \sup_{[0,1]} |f'(x)|\,, \quad \lfloor f \rfloor = \sup_{[0,1]} |f(x) + f'(x)|$$
$$\lceil f \rceil = \sup_{[0,1]} \big(|f(x)| + |f'(x)|\big)\,, \quad [f] = |f(0)| + \sup_{[0,1]} |f'(x)|\,.$$

(a) Prove that one of them is not a norm, and the remaining three are equivalent.

(b) Prove that $C^1[0,1]$ endowed with the above norm(s) is complete.

(c) Is this space complete with respect to the C^0-norm, $\|f\| = \sup_{[0,1]} |f(x)|$? Justify your answer.

6. Prove that the following are bounded linear functionals, and compute their norms:

(a) $f \mapsto \sum_{n=1}^{\infty} \frac{1}{n^2} f(\frac{1}{n})$ on $C[0,1]$ (the space of continuous functions on $[0,1]$ with the supremum norm);

(b)$f \mapsto f(0) + f'(\frac{1}{2})$ on $C^1[0,1]$

(use any of the equivalent norms on $C^1[0,1]$ from the previous problem).