Math 23b	**Quiz 3**	**April 16, 2008**

Your name: ________________

Write clearly, using complete sentences. If necessary, use additional sheets of paper with your name on them.

1. [10 points] Let A and B be two subsets of $\mathbf{R}$ which are nonempty and bounded from above, and assume that $\sup A < \sup B$ (here sup stands for the least upper bound). Prove that there exists $b \in B$ which is an upper bound for A.

   Since $\sup A$ is strictly less than the least upper bound for B, $\sup A$ is not an upper bound for B, therefore there exists $b \in B$ which is bigger than $\sup A$. Since $\sup A$ is an upper bound for A and $b > \sup A$, this b is also an upper bound for A. (Exercise: write a proof by contradiction.)

2. [15 points] Let $\langle a \rangle$ be a convergent sequence of real numbers with $a_n \geq 0$ for all $n \in \mathbf{N}$. Using the definition of limit:

(a) if $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} \sqrt{a_n} = 0$;

Since $\lim_{n\to\infty} a_n = 0$, for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $n \geq N \Rightarrow |a_n| < \epsilon^2$, and the latter is equivalent to $\sqrt{a_n} < \epsilon$.

(b) more generally, if $\lim_{n\to\infty} a_n = L$, show that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$. [Hint: when $L > 0$, write $\sqrt{a_n} - \sqrt{L}$ as $\dfrac{a_n - L}{\sqrt{a_n} + \sqrt{L}}$.]

Assume that $\lim_{n\to\infty} a_n = L > 0$. Then for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $n \geq N \Rightarrow |a_n - L| < \epsilon\sqrt{L}$. Then for those n,

$$|\sqrt{a_n} - \sqrt{L}| = \left| \frac{a_n - L}{\sqrt{a_n} + \sqrt{L}} \right| < \frac{\epsilon\sqrt{L}}{\sqrt{L}} = \epsilon\,.$$