Math 326a, Fall 2006, Problem Set # 2

The Box Principle

1. Each of 102 students in a school is a friend of at least 68 others. Prove that among them one can choose four who have the same number of friends. (Assume that friendship is a symmetric relation.)

2. Suppose that a quadratic polynomial $ax^2 + bx + c$ is equal to the fourth power of an integer for all positive integers x. Prove that $a = b = 0$.

3. 2000 real numbers are written in a row. Prove that it is possible to choose a nonempty subset consinting of adjacent numbers (maybe of just one number) such that the sum of numbers from this subset differs from an integer by no more than 1/1000.

4. An open subset U of $[0, 1]$ has the property that for any $x, y \in U$, the distance between x and y is different from 1/10. Prove that the measure (length) of U is not greater than 1/2.

5. 50 segments are given on a straight line. Suppose that no point on the line belongs to more than 7 of the segments. Prove that one can find 8 of them which are pairwise disjoint.

6. Prove that for any $n \in \mathbb{N}$ there exists a Fibonacci number divisible by n.

7. 33 rooks are placed on an 8×8 chessboard. Prove that one can choose 5 of them which are not attacking each other.

8. Let A and B be two $n \times n$ matrices with integer entries such that

$$\{A, A \pm B, \ldots, A \pm nB\} \subset \mathrm{GL}_n(\mathbb{Z})$$

(recall that $\mathrm{GL}_n(\mathbb{Z})$ stands for the group of invertible $n \times n$ matrices with integer entries whose inverses have integer entries). Prove that $A + kB \in \mathrm{GL}_n(\mathbb{Z})\ \forall\, k \in \mathbb{Z}$.

9. Prove that for any $\alpha \in \mathbb{R}$, $Q > 1$ and $k \in \mathbb{N}$ there exist at least k distinct pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that $|q\alpha - p| < 1/Q$ and $q \leq kQ$.

10. Let S be a (Lebesgue measurable) subset of $\mathbb{R}^n$ with volume strictly greater than $k \in \mathbb{N}$. Show that it is possible to translate S so that it covers at least $k + 1$ points of $\mathbb{Z}^n$.