The Consistency of the Generalized Continuum Hypothesis

The Generalized Continuum Hypothesis

The Generalized Continuum Hypothesis, was first hypothesized by Georg Cantor, and has mystified mathematicians and philosophers alike ever since. This uncanny statement is the following:

For any infinite sets x and P(x) (P(x) is the power set of x), if $x \leq y \leq P(x)$, then either y=x or y=P(x). In other words, for any set x and its power set (P(x)), there is no set whose cardinality is strictly in between x and P(x).

In the following pages we will demonstrate that the Generalized Continuum Hypothesis is consistent with the Zermelo-Frankael axiom system. That is, we will prove that it is impossible to disprove the Generalized Continuum Hypothesis (GCH) using the Zermelo Frankel (ZF) axiom system. This requires a development consisting of quite a few theorems that may seem initially irrelevant. But the reader will no doubt be awestruck when this collection of theorems fits together to prove this beautiful result. I will use the development given by Raymond M. Smullyan and Melvin Fitting in *Set Theory and the Continuum Problem*, (Clarendon Press, Oxford 1996).

This consistency of GCH is the first step to proving the *independence* of GCH, the fact that GCH cannot be proved or disproved in ZF. While here we give Godel's proof that GCH cannot be disproved in ZF, Paul Cohen has shown GCH cannot be proved in ZF either! The combination of these two theorems produces the interesting phenomenom of independence.

The independence of GCH is a beautiful example of Godel's famous incompelteness theorem. In this remarkable theorem Godel showed that given any axiomatic system, there exists a statement that cannot be proven or disproven in the system, that is, no system is complete. Thus, even if we could add an axiom to ZF that could be used to prove or disprove the Generalized Continuum Hypothesis, we could find another statement that is independent of this new axiom system.

The reason GCH provides such a vauleable example of Godel's theorem is that it is by no means trivial. On the contrary, there is nothing about GCH that intuitively suggests that it is independent. In other words perfectly "normal" statements such as the infinity of primes or the Pythagorean theorem could have potentially been independent. The fact that these independent statements not only exist, but hide behind the faces of perfectly reasonable questions is quite profound. This subject demands a willingness to enter the quest not only for the knowable, but for the knowably unknowable, a pursuit which will no doubt give the reader quite a thrill. So without further ado, let us begin.

Some Terminology

Definition 1

A **RELATIONAL SYSTEM** is a class A corresponding to a relation R on A. We will denote a relational system by (A,R).

Definition 2

Given (A,R), and an element a in A, we let a* be the class of all x in A such that xRa holds (meaning, our relation R exists between x and a). We call the elements of a* the **COMPONENTS** of A.

Definition 3

An **INITIAL ELEMENT** of (A,R) is an element x in A, such that for all y in x*, y is not in A.

So to check if x is an initial element, we consider all the elements y of A such that yRx. If any of these elements are in A, x is not an initial element. If none of these elements are in A, x is an initial element.

Definition 4

(A,R) is **WELL FOUNDED** if every non-empty subclass B of A contains an initial element.

Mostowski Shepherdson Mappings

We now proceed to prove an important theorem that generalizes the principle of induction. Unsurprisingly this theorem is called the generalized induction theorem.

Theorem 1 (Generalized Induction)

Suppose (A,R) is well founded and P is some property. To prove that P holds for all x in A it is sufficient to prove the following: For all x, if P holds for all the components of x then P holds for x.

Proof

Assume that for all x in A, if P holds for all the components of x, P holds for x. Also assume that (A,R) is well founded. We wish to prove that P holds for all x in A. We give a proof by contradiction.

Assume that for some element in A, P does not hold. Then the subclass B of A of all elements of A for which P does not hold is non-empty. Then, because A is well founded, B contains an initial element, z. So z is in B, but all the components of z are not in B (by definition of initial element). But since all elements for which P does not hold are in B, and the components of z are *not* in B, then P holds for all the components of z. But by hypothesis, if P holds for all the components of z, P holds for z! But z is in B so P does not hold for z. Contradiction.

Thus, P holds for all elements of A.

Definition 5

Given (A,R), for any subset x of A we define **THE FUNCTION p(x)** as the class of all elements a of A such that a* is a subset of x. Thus a is in p(x) if and only a* is a subset of x.

Definition 6

(A,R) is **EXTENSIONAL** if x*=y* implies x=y.

Definition 7

A relational system (A,R) is **PROPER** if for every a in A, a* is a set.

The above definitions, although somewhat cumbersome, are necessary to prove a useful lemma. Another vital ingredient is the axiom of substitution. This powerful axiom was the eighth and last axiom to be added to what is currently known as ZF.

The Axiom of Substitution

If we take a set x and substitute each element y of x with F(y), where F is any function, the resulting class is a set.

A simple corollary, which is equivalent to the axiom of substitution states that any class that can be put into a one to one correspondence with a set is a set. We shall treat this corollary as equivalent to the axiom of substitution and use the term “axiom of substitution” to mean either the axiom itself or the corollary.

Theorem 2

If (A,R) is extensional and proper, then for every subset x of A, p(x) is a set.

Proof

Suppose (A,R) is extensional and proper. Now let h(a) = a*. Because (A,R) is proper, a* is a set. Because (A,R) is extensional h(a) is a one to one function. Now, by definition of p(x), a is in p(x), if and only if h(a) is a member of P(x) (that is, h(a) is a subset of x).
So let us take an element a of p(x). What happens if we apply h to a? We get an element of P(x)! Thus h maps every element of p(x) onto an element in P(x). So h maps the class p(x) onto a subset of P(x) in a one to one manner. P(x) is a set by ZF, so any subset of P(x) is also a set.
Then because we are mapping the class p(x) onto a set in a one to one manner, p(x) is a set by the axiom of substitution.

Definition 8

Ã-RANK is defined only over relational systems (A,R) such that for every subset x of A, p(x) is a set. Ã-rank is defined by the following ordinal sequence:
$Ã_0$ = the empty set
$Ã_{á+1} = p(Ã_{á})$ (á+1 is a successor ordinal)
$Ã_{ë} = U_{á<ë} Ã_{á}$ (ë is a limit ordinal)
An **element x of A has Ã-rank** if it is in some $Ã_{á}$, in which case we define its Ã-rank to be the first ordinal á such that x is in $Ã_{á+1}$.

Properties of Ã-rank

Although the following properties are not difficult to verify it is somewhat tedious to do so. They are mostly consequences of general properties that apply to all ordinal hierarchies. We therefore list them here without proof:
P$_1$ $á \leq â$ if and only if $Ã_{á} \leq Ã_{â}$.
P$_2$ x is in $Ã_{á}$ if and only if x is in $Ã_{â+1}$ for some $â < á$.
P$_3$ x is in $Ã_{á}$ if and only if x has Ã-rank less than á.

P$_4$ x has Ã-rank if and only if x is a member of Ã$_{á+1}$- Ã$_á$.
P$_5$ For any subclass B of A and any ordinal á, B is a subset of Ã$_á$ if and only if all elements of B have Ã-rank less than á.
P$_6$ For any x in A, x* is a subset of Ã$_á$ if and only if x hasÃ-rank $\leq$ á.
P$_7$ For any element x of A, if x* is a set and every element of x* has Ã-rank, then x has Ã-rank.

Theorem 3

Suppose that (A,R) is a proper well founded relational system such that for every subset x of A, p(x) is a set. Then every element of A has Ã-rank.

Proof

Because for every subset x of A, p(x) is a set, the notion of Ã-rank is well defined. We wish to apply the general induction theorem to show that every element of A has Ã-rank. To do so we must prove that *if every element of x* has Ã-rank, then x also has Ã-rank*. So assume that every element of x* has Ã-rank. We need to show that x also has Ã-rank. A glance at P$_7$ will reveal that this particular property will be very useful. P$_7$ states that *if x* is a set, and every element of x* has Ã-rank, then x has Ã-rank*. The second hypothesis (every element of x* has Ã-rank) is fulfilled by our assumption, so we just need to show that x* is a set. This is easy because we assumed that (A,R) was proper and so x* is a set by definition. Thus both hypotheses of P$_7$ are fulfilled and so we can conclude every element of A has Ã-rank.

Theorem 4

If (A,R) is a proper, extensional, and well founded system, then every element of A has Ã-rank.

Proof:

Theorem 2 and Theorem 3

The next theorem we will prove is important for a key result known as the Mostowski Shepherdson mapping theorem. But first we must introduce some notation:

Definition 9

Let A be a set, with elements a$_1$,a$_2$, a$_3$….. and let f be a function defined on elements of A. By **f''(A)** we mean the set containing f(a$_1$), f(a$_2$)….f(a$_n$)…. In other words f''(A) is the image of A under F.

Theorem 5 (Generalized transfinite recursion theorem)

Let (A,R) be a relational system such that every element of A has Ã-rank. Then for every function g defined on V (the class of all sets) there is a function F defined on A such that for every x in A, F(x) = g(F''(x*)).

Proof

Let us define the following recursive function:
a) f$_0$ = the empty set
b) i) *if x is in Ã$_á$, f$_{á+1}$(x) = f$_á$(x).*
  ii)*If x is in Ã$_{á+1}$-Ã$_á$, f$_{á+1}$(x)= g(f$_á$''(x*)).*
c) f$_ë$(x) (where ë is a limit ordinal) = U$_{á<ë}$ f$_á$(x)

Since $f_{á}$ is a subset of $f_{á+1}$, this is an ordinal hierarchy, and thus for $á \leq â$, $f_{á}$ is a subset of $f_{â}$. So, for each x in $Ã_{á}$, $f_{á}(x) = f_{â}(x)$.
We now define F(x) to equal $f_{â}(x)$, where â is the first ordinal such that x is in $Ã_{â}$. Thus for any á greater than or equal to â, if x is in $Ã_{á}$, then $F(x) = f_{á}(x)$. But since x is in $Ã_{â}$, and $â \leq á$, $f_{â}(x) = f_{á}(x)$, and thus $F(x) = f_{á}(x)$.
We now must prove that F(x) = g(F''(x*)).
Let á be the Ã-rank of x. This means that x is in $Ã_{á+1}$-$Ã_{á}$. We know (by definition of Ã-rank) that all the components of x are in $Ã_{á}$. So x* is a subset of $Ã_{á}$. Then every element y of x* is an element of $Ã_{á}$. So $F(y) = f_{á}(y)$ (because y is an element of $Ã_{á}$). Now instead of looking at the individual elements of x* we can look at the image of x* as a whole.
Thus, $F''(x^*) = f_{á}''(x^*)$.
Applying g to both sides:
$g(F''(x^*)) = g(f_{á}''(x^*))$.
Now it remains to show that $g(f_{á}''(x^*)) = F(x)$. Well, we know that the Ã-rank of x is á. x is therefore in $Ã_{á+1} - Ã_{á}$. So $f_{á+1}(x) = g(f_{á}''(x^*))$. But since á+1 is the first ordinal such that x is in $Ã_{á+1}$, $F(x) = f_{á+1}(x) = g(f_{á}''(x^*)$.
Q.E.D

Theorem 6

If (A,R) is proper, extensional, and well founded, then for any function g defined on all sets there is a function F on A such that for all x in A, F(x) = g(F''(x*)).

Proof

Because (A,R) is proper, extensional, and well founded, we know by theorem 4 that every element of A has Ã-rank. Then, by theorem 5, for any g, there exists a function F on A such that for all x in A, F(x) = g(F''(x*)).
Q.E.D.

We now consider an important example of this theorem, where g is the identity function. We thus have the following:

Theorem 7

If (A,R) is proper, extensional, and well founded, then there is a function F on A such that for all x in A, F(x) = F''(x*).

Definition 10

A function F satisfying the conclusion of theorem 7, that is, that F(x) = F''(x*) for all x in A, is called a **MOSTOWSKI-SHEPHERDSON MAP** for (A,R).

Definition 11

A class A is called **TRANSITIVE** if every element of A is a subclass of A. A somewhat more clumsy way of saying this is that all the elements of elements of A, are themselves elements of A. Or: "If x is in y, and y is in A, then x is in A."

Definition 12

An **ISOMORPHISM** between two relational systems: (A_1,R_1) and (A_2,R_2), is a one-to-one function (F) from A_1 onto A_2, such that xR_1y if and only if, $F(x)R_2F(y)$. Two relational systems: (A_1,R_1) and (A_2, R_2) are called **ISOMORPHIC** if there exists some isomorphism between (A_1, R_1) and (A_2, R_2)

Theorem 8 (Mostowki-Shepherdson mapping theorem)

Suppose (A,R) is a relational system that is proper, extensional, and well founded. Then:

a) there is a Mostowski-Shepherdson map F for (A,R)
b) (A,R) is isomorphic to (F''(A),å) under F (å is the relation: *inclusion*).
c) F''(A) is transitive.

Proof

a) It is obvious by theorem 7 that there exists a Mostowski-Shepherdson map because (A,R) is proper, extensional and well founded.

b) i)Next, in order to prove that F is an isomorphism we must first prove that F is one to one. That is, we must prove that for all elements x and y in A, F(x) = F(y) implies that x=y.

We will use generalized induction (theorem 1) to prove this. First, take an element x of A. We shall call x *good* if for all elements y of A, F(x) = F(y) implies that x=y. To use generalized induction, we assume that all elements of x* are good. It remains to show that x is good. That is we must show that F(x)=F(y) implies x=y. Well, since (A,R) is extensional it suffices to show that x*=y*, for this automatically implies that x=y. Thus, using the assumption that the elements of x* are good, we will prove that F(x)=F(y) implies that x*=y*. So assume that F(x)=F(y).

1) First we show that x* is a subset of y*. Let z be any element in x*. We will show that z is an element of y* as well. So consider the set F''(x*) (that is the image of x* under F). Because z is an element of x*, F(z) is an element of F''(x*). But because F is a Mostowski Shepherdson map, F''(x*)=F(x). By assumption, F(x)=F(y), and also F(y)=F''(y*). So F(z) is also an element of F''(y*). But if F(z) is in the image of y*, then there must be some w in y* such that F(z)=F(w). But since z is an element of x*, z is good, and thus this implies that z=w. Thus, z is an element of y*, and so x* is a subset of y*.

2) Next we show that y* is a subset of x*. The proof of this is very similar to part 1. So this time let's start with w, an element of y*. We must show that w is an element of x*. Because w is in y*, F(w) is an element of F''(y*), which, as we have shown in part one, is the same as F''(x*). (This is because F is a Mostowski Shepherdson map, that thus F''(y*)=F(y) which by assumption =F(x) which equals F''(x*)). So F(w) is an element of F''(x*). Thus, F(w)=F(y) for some y in x*. But if y is in x*, y is good, and thus by the inductive hypothesis, w=y, and thus w is an element of x*. So y* is a subset of x*.

By 1) and 2) we get that x*=y*, and thus that x=y. We have thus shown that F is a one to one function.

ii) The next step in proving that F is an isomorphism from (A,R) onto (F''(A),å) is proving that xRy is true if and only if F(x) å F(y) (F(x) is an element of F(y)).

So assume that xRy. Then by definition x is an element of y*. So F(x) is an element of F’’(y*) = F(y). Thus, F(x) is an element of F(y). Now to prove the converse, Assume that F(x) is an element of F(y). Then F(x) is an element of F’’(y*), so F(x)=F(z) for some z in y*, but because F is one to one, x=z. Thus x is in y*, and thus xRy. So F is an isomorphism.

c) Lastly we must show that F’’(A) is a transitive class. This means that we need to prove that every element of F’’(A) is also a subclass of F’’(A). Well, let x be an element of F’’(A). We are to show that x is also a subset of F’’(A). Since x is an element of F’’(A), x= F(z) for some z in A. Because F is Mostowski-Shepherdson, x also equals F’’(z*). Now we know that z* is a subset of A, so F’’(z*) is a subset of F’’(A). But x equals F’’(z*), so x is also a subset of F’’(A).

Q.E.D.

Definition 13

Two classes A and B are called **å-ISOMORPHIC** if the two corresponding relational systems (A,å) and (B,å) are isomorphic. An **å-ISOMORPHISM** is an isomorphism F from (A,å) onto (B,å).

Theorem 9

Every extensional, well founded class is å-isomorphic to a transitive class. More specifically, for any well founded, extensional class A, there is a map F with domain A such that for every x in A, F(x) = F’’(x   A), and this F is an å-isomorphism from A onto a transitive class.

Proof

We will apply theorem 8, letting R be the inclusion relation (å). The relational system (A,å) is proper because given any element a of A, the elements of a will form a set. Thus, (A,å) is extensional, well founded and proper, and there then must exist a Mostowski-Shepherdson map for (A,å). Then by theorem 8, F is an å-isomorphism from A onto a transitive class (in particular: F’’(A)). Now, in the relational system (A,å), x* is the set of all elements of A that are also elements of x, in other words: (x   A). So instead of writing the Mostowski Shepherdson map as a function F such that F(x) = F’’(x*), we can say that F(x)= F’’(x   A), and as we have shown before F maps A onto the transitive class F’’(A).

Q.E.D.

Theorem 10

Suppose that A is well founded and extensional, that T is a transitive subclass of A, and that F is a Mostowski-Shepherdson map on (A,å). Then for each element x of T, F(x)=x.

Proof

Assume the hypothesis. Now let x be an element of T. Because T is transitive x is also a subset of T, which is a subset of A. So x is a subset of A, and thus x = x   A. So F’’(x)=F’’(x   A). But F’’(x   A) = F(x) by the previous theorem. Thus we have that F(x)=F’’(x). We will now do generalized induction to show that F(x)=x. Since we are dealing with (A,å), the components of x, are all elements y that are members of x. So assume that for all y in x, F(y)=y. Now consider F’’(x), that is, the image of x. F’’(x) is the set of all elements F(y) where y is in x. But F(y) is just y! So this set is identical to x! Thus F''(x)=x. But we have shown that F''(x)=F(x), so F(x)=x. By generalized induction, this is true for every element of T.

The M.S.T.V Theorem

Some Preliminaries

At this point the reader may be wondering why we created these "arbitrary" Mostowski-Shepherdson maps, and what all this dribble has anything to do with the continuum hypothesis. It will all become abundantly clear soon and at this juncture, we have the opportunity to make some of the important bridges that will connect us from these abstract mappings to the problem at hand.

We will be dealing with first order logic, that is, a logical system based on the primitive connectives of "not," "and," as well as "there exists." Our variables will be called $x_1, x_2, x_3 \ldots\ldots\ldots x_n \ldots\ldots\ldots\ldots$

Before we proceed we must make an attempt at defining truth. To avoid lengthy philosophical problems we will limit our definition to truth in a given relational system.

Definition 14

An **ATOMIC FORMULA** in a system (A,R) is an expression xRy, where x and y are any individual variables.

For any formula ö, -ö (not ö) is a formula.

For any formulas ö and ø, the expression (ö ø) (ö and ø) is a formula.

For any formula ö and any variable x, the expression (x)ö (there exists an x such that ö) is a formula.

Definition 15

PROPER SUBFORMULAS are defined as follows:

Atomic formulas have no proper subformulas.

The proper subformulas of ~ö are ö and the proper subformulas of ö.

The proper subformulas of (ö ø) are ö, ø, and the proper subformulas of ö and the proper subformulas of ø.

The proper subformulas of (x)ö are ö and the proper subformulas of ö.

The **SUBFORMULAS** of ö are ö and its proper subformulas.

Definition 16

The **DEGREE OF A FORMULA** is the number of occurrences of the symbols ~, , . Thus:

Atomic formulas are of degree 0.

If ö is of degree n, then ~ö is of degree n+1.

If ö is of degree n and ø is of degree m than (ö ø) is of degree n+m.

If ö is of degree n, then (x)ö is of degree n+1.

Definition 17

A **SENTENCE** is a formula with no free variables.

Definition 18

Consider a relational system (A,R) where A is a set. By an **A-SENTENCE** we mean a sentence whose constants are all members of A.

An atomic sentence aRb (a and b are elements of A) is called **TRUE** if and only if aRb holds.

For any sentence ö, with constants in A, the sentence –ö is true in (A,R) if an only if it is not the case that ö is true in (A,R).

For any sentences ö and ø, both with constants in A, (ö ø) is true if and only if both ö is true and ø is true.

For any sentence ö with constants in A, (x)ö is true if and only if there is at least one element a of A such that ö(a) is true in (A,R).

Induction over the Degree of a Formula

To show that a given property holds for all A-formulas it suffices to show the following:

a) The property holds for all atomic A-formulas.
b) If the property holds for ö it holds for ~ö.
c) If the property holds for both ö and ø, then it holds for (ö ø).
d) If the property holds for ö, then for every variable x, the property holds for (x)ö.

Definition 19

We will say that sentence X with constants in A is **TRUE OVER A** if and only if it is true in the relational system (A,å).

Definition 20

Given a set A and subset B, we will say that **B REFLECTS A** if for every sentence X whose constants are all in B, X is true over B if and only if X is true over A. **B REFLECTS A WITH RESPECT TO ö** if for all constants b_1,….b_n of B, the sentence ö(b_1….b_n) is true over B if and only if it is true over A. **B COMPLETELY REFLECTS A WITH RESPECT TO ö** if B reflects A with respect to all subformulas of ö.

Definition 21

Let B be a subset of A and let ö be a formula whose constants are all in B. B is **A-CLOSED WITH RESPECT TO ö** if for every subformula of ö of the form (x)ø(x,y_1….y_n) where the variables of ø are x,y_1……..y_n, and for all elements b_1……b_n of B, if there is some element a of A such that the sentence ø(a,b_1……..b_n) is true over A, then there is some b in B such that ø(b,b_1…..b_n) is true over A.

Definition 22

B is **A-CLOSED** or **HENKIN-CLOSED** if B is A-closed with respect to every formula ö whose constants are all in B.

We are finally prepared to begin proving a sequence of theorems that will lead us to an important result.

Theorem 11

If B is A-closed with respect to ö, then B completely reflects A with respect to ö.

Proof

We will call a formula ö whose constants are in B *good* if B reflects A with respect to ö. Now, assume that B is A-closed with respect to ö.

We will do induction on the degree of the formula to prove that B completely reflects A with respect to ö.

a) An atomic formula is simply an expression that states a relation between two elements. In this sense it is independent of its surroundings. It is thus intuitively obvious that an atomic formula's truth over B and over A would be equivalent. More specifically, an atomic formula has no variables and its truth value is therefore absolute. We can thus say that an atomic formula is good.

b) The negation of a good formula is good for the same reason. If a formula's truth value is independent of its setting in A or B, its negation in our binary truth system of true and false is simply the opposite.

c) Similarly, the conjunction of two good formulas is good.

d) Finally we must show that if ø(x,y_1…..y_n) is good, then (x)ø(x,y_1…..y_n) is good as well. So assume that ø(x,y_1…..y_n) is good. We must show that (x)ø(x,y_1…..y_n) is true over A if and only if it is true over B.
   - Suppose that (x)ø(x,y_1…..y_n) is true over B. Then for some b in B, ø(b,b_1,…..b_n) is true over B. But since ø(x,y_1…..y_n) is good by our inductive hypothesis, ø(b,b_1….b_n) is also true over A. Also b is an element of A, since B is a subset of A. So, (x)ø(x,y_1…..y_n) is true over A as well.
   - Now, suppose that (x)ø(x,y_1…..y_n) is true over A. We must show that it is true over B. Note that we have still not used the hypothesis of our theorem, but it is at this juncture that the hypothesis will be used. By hypothesis, B is A-closed with respect to ö. This means, that if ø is a subformula of ö, and there exists an element a in A such that the sentence ø(a,b_1……b_n) is true over A, then there exists an element b in B such that ø(b,b_1……b_n) is true over A. Now, since we are supposing that (x)ø(x,y_1…..y_n) is true over A, we are guaranteed that there exists some a in A such that ø(a,b_1……b_n) is true over A. Thus there exists an element b in B such that ø(b,b_1……b_n) is true over A. But by our inductive hypothesis ø is good, so if ø(b,b_1….b_n) is true over A it is also true over B. Thus, (x)ø(x,y_1…..y_n) is true over B as well.

The following theorem is an immediate corollary of theorem 11.

Theorem 12

If B is Henkin closed with respect to A, then B reflects A.

The reader will recall proving the Mostowkski-Shepherdson theorem earlier. The M.S. theorem is actually one of the two parts of the M.S.T.V (Mostowski Shepherdson Tarski Vaught) theorem which is necessary for the proof of the consistency of the Generalized Continuum Hypothesis. We will now prove the second piece needed to discuss M.S.T.V., the unsurprisingly named Tarski-Vaught Theorem.

Theorem 13 (Tarski Vaught)

For any set A that can be well ordered, any infinite subset A_0 of A can be extended to (is a subset of) a subset B of A of no higher cardinality than A_0 such that B reflects A. (In particular, any denumberable subset A_0 of A can be extended to a denumerable subset B of A such that B reflects A.)

Proof

Suppose that A is an infinite and well ordered set with cardinality c, and let A_0 be a subset of A of cardinality c.

Now, consider any subset S of A. We will define S' as follows:

First we look at all the formulas with constants in S of the form (x)ö(x) that are true over A. Then for each formula ö such that (x)ö(x), we take the first element a in A (in the well ordering of A) such that ö(a) is true over A. Adjoining all such elements a to S we get S'.

Since the cardinality of A is c, the cardinality of S is at most c. If the cardinality of S were c, then there would be no more than c constants of which to construct these formulas. Since formulas are simply manipulations of these constants, a countable number of variables and a finite number of symbols, the cardinality does not change when considering the number of formulas with constants in S. Thus, there are c formulas with constants in S.

We now consider the following sequence:

A_0, A_1, A_2…….A_n, A_{n+1}……….

Where $A_{n+1} = A_n$'

Let B be the union of all the A_n s.

Each A_n then has cardinality c and so the union of all the A_n s has cardinality c.

Now, take any formula ø with constants in B. There are finitely many constants in ø (by definition of a formula) and since A_n is a subset of A_{n+1} (because we get A_{n+1} by adjoining certain elements to A_n), then all of the constants in ø lie in some A_n. Now if it happens to be that (x)ø(x) is true over A, then there is some element a in A_{n+1} such that ø(a) is true over A. And thus, a is in B.

So B is Henkin closed with respect to A.

By the previous theorem, this implies that B reflects A.

Theorem 14

Let A be a well orderable infinite set that is extensional with respect to å. Then any infinite subset A_0 of A can be extended to an extensional subset B of A of no higher cardinality than A_0 such that B reflects A.

Proof

The proof of this theorem is almost identical to the previous one except that we slightly modify our sequence A_0, A_1…..A_n, A_{n+1}…

This time, A_{n+1} contains the same elements as A_n' along with the following:

Because A is extensional, for every pair of elements x, y in A, there exists an element z in A such that z belongs to either x or y but not both. For each pair (x,y) we add in the first such z in the well ordering of A to A_{n+1}. The cardinality of A_{n+1} is still c because there are c pairs (x,y) and the cardinality of A_n' is c. So adjoining the z's to A_n' results in what we shall call here A_{n+1}, a set of cardinality c.

Again, let B be the union of all the A_n. Note that B has cardinality c because it is the union of sets of cardinality c.
By the same argument as before, B reflects A so it remains to show that B is extensional. Well, for any pair x,y in A there exists an A_n such that both x and y are in A_n. Then, there exists a z that is a member of x or y but not both in A_{n+1}. Thus B is a subset of A with cardinality c that is extensional and reflects A.

Dealing with Classes instead of Sets:

<u>Definition 23</u>:
A set A is **DEFINABLE** if there exists a formula ö(x) such that a is a member of A if an only if ö(a) is true over A.

We are now prepared to deal with the class version of the Tarski Vaught theorem. First, it is important to realize that any sentence can be coded by a unique set. A sentence is, after all, only a finite string of symbols and it is not difficult to associate each such string with a set. We will refer to a sentence as a sentence or a set interchangably.
Now, for any class K, let T(K) be the class of all sentences with constants in K which are true over K. We ask the following question:
Does there exist a formula ö(x) such that the sentence ö(a) is true over K if an only if a is a sentence with constants in K which is true over K? In other words, is T(K) ***definable*** over K?
The answer to this question turns out to be no if K is not a set.
However, consider a different class, $T_n(K)$, which is the class of all formulas with degree less than or equal to n with constants in K which are true over K. We can prove by induction that each of these T_n is definable, even though their union T(K) is not.
The class $T_0(K)$ is simply the class of all sentences a å b, (where å symbolizes the relation of the system we are dealing with) where a and b are elements of K and it is true that a å b.
Now assume that $T_n(K)$ is definable. We will show that $T_{n+1}(K)$ is definable as well.
$T_{n+1}(K)$ is the class that contains

- All the elements of $T_n(K)$
- Sentences of the form ~X (where X is a sentence of degree n),
- Sentences of the form X Y (where X and Y are of degree n)
- Sentences of the form (x) ö(x) (where ö(x) is a formula such that for some set a, ö(a) is a sentence of degree n).

We have just defined $T_{n+1}(K)$.
Thus if K is definable over V, then for all n, $T_n(K)$ is definable over V.
The purpose of this rigor was to allow us to talk freely about the notions of reflection with respect to ö, and being K-closed with respect to ö, while working in classes.
Because we have shown that for all n $T_n(K)$ is definable, we can conclude that the following sentences are definable (where K is a particular class, B is a set and ö is a particular formula with constants in K):

1) B reflects K with respect to ö and all its subformulas.
2) B is K closed with respect to ö.

Notice, that if we did not limit our discussion to a particular formula ö and its subformulas we would be treading on the forbidden territory of T(K), which is not definable. This is because when saying that B reflects K with respect to *any* formula ö, we are inadvertently referring to the class of all formulas with constants in K.

Theorem 15

Suppose K is well orderable and A_0 is an infinite subset of K of cardinality c. Then for each formula ö:

1) A_0 can be extended to (i.e: is a subset of) a subset B of K of cardinality c such that B completely reflects K with respect to ö.
2) if also K is extensional, then A_0 can be extended to an extensional subset B of K of cardinality c such that B completely reflects K with respect to ö.

We omit the proof of this theorem, as we have justified the use of classes in this case, and the proof is virtually identical to the one with sets. However, in the construction of the sequence A_0, A_1,……A_n, A_{n+1}…… we don't consider all formulas with constants in A_n (because this class is undefinable!), but only formulas of the form (x) ø (x, a_1…..a_k) where ø(x, y_1…..y_k) is a proper subformla of ö, and a_1…a_k are constants in A_n.

Finally, we present the theorem which will tie in everything we have seen thus far, into a very beautiful and important result known as the Mostowski Shepherdson Tarski Vaught Theorem, or the M.S.T.V theorem for short.

Theorem 16 (M.S.T.V)

Let K be an extensional, well founded class that can be well ordered, and let X be a sentence that is true over K. Then any infinite transitive subset A of K is a subset of some transitive set T of the same cardinality as A, such that X is true over T.

Proof

Let A be an infinite subset of K of cardinality c. We will prove that A is a subset of some transitive set T of cardinality c such that X is true over T. First, we know by theorem 15, that since K is well ordered and extensional, A is a subset of some extensional subset B of cardinality c such that B completely reflects K with respect to X. So since X is true over K, X is true over B as well. Now, let us review the definition of well founded. A class is well founded, if every non-empty subclass of the class contains an initial element. We know that K is well founded so every non-empty subclass of K contains an initial element. We also know that B is a subset of K, so B must also be well founded, since every non-empty subclass of B is also a non-empty subclass of K and therefore contains an initial element. We now have that A is a subset of B where X is true over B, B is of cardinality c, B is extensional and B is well founded. It as at this point that Mostowski-Shepherdson mappings become important. By theorem 9, every extensional well founded class is å-isomorphic to a transitive class. Call this transitive class T. Since T is å-isomorphic to B, and X is true over B, X is true over T as well. Also T is of cardinality c since it is å-isomorphic to B. Thus, it remains to show that A is a subset of T. In the proof of theorem 9, the map which creates this isomorphism between B and T is actually

a Mostowski-Shepherdson map. So we can now apply theorem 10 which states that this map will carry each element of B over to itself. Thus B is a subset of T. A is a subset of B, so A is also a subset of T which have shown to be a transitive set of cardinality c such that X is true over T.
Q.E.D.

Constructibility and its Implications

Our first task is to clearly describe the notions of definability and constructiblity. As we shall see later on, the question of what kind of sets are constructible, is not only important but even somewhat mystical.

Definition 24
Given a set A, we denote by ***F*(A)** the set of all subsets of A that are definable over A. We now define the following recursion:
L_0 = the empty set
$L_{á+1} = F(L_á)$
$L_ë = U_{(á<ë)} L_á$ (ë is a limit ordinal).
We let **L** be the union of all the $L_á$. The elements of L are called **CONSTRUCTIBLE**. For any constructible set x we mean by its **ORDER** the first ordinal á such that x is a member of $L_{á+1}$.

Next, we must define what it means for a **sentence** to be **ABSOLUTE**. To do this, consider a class K and sentence X. X is either true over K or it is false over K. When we say that X is true over K we interpret the phrase “for all x” to mean “for all x in K,” and “there exists an x,” to mean “there exists an x in K.” Now, X may or may not be true over V as well. That is, if instead of “for all x in K” we said “for all x in V” (or “for any set x” since V is the class that contains ALL sets), X may or may not change it truth value. If the truth value of X over K is equivalent to its truth value over V, we say that X is absolute over K.

Definition 25
Consider a formula $ö(x_1 \ldots x_n)$ with no constants. We say that the **formula ö is ABSOLUTE OVER K** if for any elements $a_1 \ldots a_n$ of K, the sentence $ö(a_1 \ldots a_n)$ is absolute over K. We shall say that a **class is ABSOLUTE OVER K** if it is defined over K by at least one formula that is absolute over K. Finally we shall say that a formula or class is **ABSOLUTE** if it is absolute over every transitive class K.

Theorem 17
Every ordinal á is constructible and is of order á.
Proof
$F(L_á)$ is a subset of $P(L_á)$ (the power set of $L_á$) since $F(L_á)$ contains only definable subsets of $L_á$ whereas $P(L_á)$ contains all of the subsets of $L_á$. Now we must introduce yet another recursion:
R_0= the empty set
$R_{á+1} = P(R_á)$

$R_{ë}$ = $U_{á<ë}$ $R_{á}$ (ë is a limit ordinal)

Had we chosen to further develop this recursion, which would lead us to a definition of rank, we could easily prove that each ordinal á is a member of $R_{á+1}$, but not a member of $R_{á}$. However, since this is the only place where we will need to use rank to prove the consistency of GCH, we will omit this proof.

Now, given that *F*($L_{á}$) = $L_{á+1}$ is a subset of P($R_{á}$) = $R_{á+1}$, we can write instead that $L_{á+1}$ is a subset of $R_{á+1}$ and so, for any successor ordinal á, $L_{á}$ is a subset of $R_{á}$. Similarly for limit ordinals. So if á is not a member of $R_{á}$, it is also not in $L_{á}$. Also á is in $R_{á+1}$ but is not in $R_{á,}$ and so if á is in $L_{á+1}$ it is not in $L_{á}$. Thus, it remains to show that á is in $L_{á+1}$ to prove that the order of á is á. We do this by transfinite induction. So assume that for every â<á, â is a member of $L_{â+1}$.

Since â<á, â+1≤ á. Now, these L's contain one another like Russian dolls. So â+1≤ á, implies that $L_{â+1}$ is a subset of $L_{á}$. Now, we know that â is a member of $L_{â+1}$ by our inductive hypothesis. So it follows that â is a member of $L_{á}$. We have also shown that á is not a member of $L_{á}$. It is not difficult to show that $L_{á}$ is transitive given the fact that $R_{á}$ is transitive, a fact that we will not show here. So every element of a set in $L_{á}$ is also an element of $L_{á.}$ Our next step is to show that there is no ordinal greater than á, in $L_{á}$. We can prove this by contradiction. Say there was an ordinal greater than á in $L_{á}$. Then, since $L_{á}$ is transitive, all the elements of that ordinal would also be in $L_{á}$. But if this ordinal is greater than á, then á itself would be a member of this larger ordinal! Thus, á is in $L_{á}$, but we have already shown that á cannot be a member of $L_{á}$. Contradiction. So we now know that not only is á not a member of $L_{á}$, but also that no ordinal greater than á is a member of $L_{á}$.

Finally, the formula "x is an ordinal" is absolute. In other words, if a set is an ordinal, it is an ordinal regardless of its settings. So, consider the formula "x is an ordinal" in $L_{á}$. We will define a set using this formula, and thus take every element of $L_{á}$ for which this formula is true, and throw it in the set. What set is this? Well, all ordinals less than á are in the set (because we have shown that every â is a member of $L_{á}$), but the ordinal á is not in this set. This set, then, IS the ordinal á itself! But this set is also a definable subset of $L_{á}$! So by definition of $L_{á+1}$ it is a member of $L_{á+1}$. Thus á is a member of $L_{á+1}$, has order á, and is constructible.

Q.E.D.

Definition 26

A **Ä FORMULA** is a formula that obeys the following rules:

1) Any atomic formula x å y is Ä.
2) If ö and ø are Ä, so are ~ö and ö ø.
3) If ö is Ä, then for any distinct variables x, y (x å y)ö, and (x å y)ö (where x, stands for: "for all x in y").

A class or relation is called a **Ä CLASS** is it is definable over V by a Ä formula.

Definition 27

A **Ó FORMULA** is defined by the following rules:

1) Every Ä formula is a Ó formula.
2) If ö and ø are Ó formulas, so are ö ø, and ö V ø.

3) If ö is a Ó formula, so is (x)ö.
4) If ö is a Ó formula so are (x å y)ö and (x å y)ö.

Definition 28

A formula ö(x_1…..x_n) is **ABSOLUTE UPWARDS** over a class K if for any elements a_1….a_n of K, if ö(a_1…a_n) is true over K then it is true over V.
Note: this is simply requiring only one direction of the definition of absoluteness.

Theorem 18

All Ó formulas are absolute upwards.

Proof

The proof of this theorem requires simply an induction argument on Ó formulas. It is easy to show that every Ä formula is absolute, for a Ä formula is simply a variant of the expression xRy which has nothing to do with the setting of x and y. So since a Ä formula is absolute, in particular, it is absolute upwards. Now once that we have established that a Ä formula is absolute, a simple induction argument based on the structure of Ó formulas, will show that any Ó formula is absolute as well. As we have done numerous similar inductions we omit the details.

Theorem 19

If ö and ~ö are both absolute upwards, then ö is absolute.

Proof

We give a proof by contradiction.
Assume that ö is true over V, but ö is not true over a class K. Then, ~ö is true over K. But ~ö is absolute upwards so ~ö is true over V as well. So ö and ~ö are true over V. Contradiction.

Theorem 20

Let F be a function defined on V and let F* be the function on ordinals defined from F by the following recursion:

1) F*(0) = the empty set
2) F*(á+1) = F(F*(á))
3) F*(ë) = $U_{á<ë}$ F*(á)

Then if F is Ó, F* is also Ó and F* is absolute.

Note:

We will not prove this theorem. To do so involves the somewhat cumbersome process of defining F* as a string of symbols that upon inspection is Ó. This is not exceedingly complicated; it requires only repeated use of existential and universal quantifiers.

Definition 29

A formula is **MONADIC** if it has exactly one variable. We remind the reader that every formula can be coded as a set. So we let **M(a)** be the set of codes of monadic formulas whose constants are in a.

Definition 30

We will use **Def(x,y,a)** to mean that x is the code of a formula ö whose constants are in a that defines y over a.

Theorem 21

The relation y=*F*(x) is Ó (where *F*(x) is the set of definable of subsets of x).

Proof

1) The relation: “y is a member of *F*(x) is Ó because it can be written as: (z å M(x))(Def(z,y,x).
2) The relation “y is a subset of *F*(x)” is Ó because it can be written as: (z å y) (z å *F*(x)).
3) The relation “*F*(x) is a subset of y” is Ó because it can be written as: (z å M(x)) (w å y) Def(z,w,x).
4) The relation y = *F*(x) is Ó since it can be restated as “y is a subset of *F*(x) and *F*(x) is a subset of y.” Both parts of this relation are Ó as shown in 2 and 3, and the combination of any two Ó relation with a conjunction is Ó.

Theorem 22

The function which assigns to each ordinal á the set $L_{á}$ is Ó and absolute.

Proof

We know by theorem 21 that *F* is Ó. So we can apply theorem 20, which tells us that as long as the discussed function is Ó, then F* is Ó as well. But what is F* in this case?

F*(0) = the empty set

F*(á+1) = *F*(F*(á))

F*(ë) = $U_{á<ë}$ F*(á)

But this is exactly, how $L_{á}$ is defined for any given á! So the function which assigns to each ordinal á the set $L_{á}$ is Ó.

Definition 31

Let **M(x,y)** be a Ó formula fixed for the discussion that defines the relation $L_x = y$.

Theorem 23

The formula M(x,y) is Ó and absolute.

Proof

Obvious from definition 31 and theorem 22.

Definition 32

Let **L(x,y)** be the Ó formula (z)(M(y,z)   x å z).

Theorem 24

The formula L(x,y) is absolute upwards.

Proof

Because by definition it is Ó.

We are approaching one of the most exciting parts of the subject. Our aim here is toexamine the axiom of constructibility:

The Axiom of Constructibility

Every set x is constructible. In other words (x å V) implies that (á)(x å $L_{á}$).

The importance of this axiom will be readily apparent in proving the consistency of GCH. Once we have shown that it is consistent to believe that every set is constructible, we will be only one step away from proving the consistency of GCH. How the Axiom of Constructibility is inextricably linked to the Generalized Continuum Hypothesis will be the subject of the following discussion.

Now, we have shown that the formula L(x,y) is absolute upwards. L(x,y) is in fact NOT absolute over all transitive classes, but we do have the following theorem:

Theorem 25

The formula L(x,y) and hence also the relation x å L_y is absolute over L.

Proof

We know by theorem 24 that L(x,y) is absolute upwards, so in particular it is absolute upwards over L. To prove absoluteness over L we must also prove that L(x,y) is absolute downwards. In other words, if L(x,y) is true over V it is also true over L.
So take any two elements, a and b, of V and assume that L(a,b) is true over V. Our job is to prove that L(a,b) is true over L as well.
Since L(a,b) is true over V, b is really an ordinal and a is a member of L_b. Let c = L_b. Then a is a member of c, and c is of course in L. So the sentence a å c is true over L. Also M(c, b) is true over V (by definition of c), but we have already shown in theorem 23 that M(c,b) is absolute. So M(c,b) is also true over L. So the combination M(c,b) ^ a å c is true over L. But this is the statement L(a,b)! So L(a,b) is true over L. Thus the formula L(x,y) is absolute over L.

Theorem 26

Let **Const(x)** be the formula (y)L(x,y). Const(x) is absolute upwards.

Proof

L(x,y) is Ó (definition 32), so Const(x) is also Ó, and thus absolute upwards.

Finally, our exceedingly important result:

Theorem 27

The property of being a constructible set is absolute over L.

Proof

We have already shown in theorem 26 that the formula Const(x) is absolute upwards. It remains to show that it is absolute downwards, that is if Const(x) is true over V, then Const(x) is true over L. So assume that Const(x) is true over V. Then for some ordinal á, x å $L_{á}$. Thus, L(x,á) is true over V. Also, á å L since all ordinals are constructible by theorem 17. So (y)L(y,á) is true over L. But this is the formula of Const(x). So if Const(x) is true over V it is also true over L. Thus the property of being a constructible set is absolute over L.

Theorem 28

The axiom of constructibility is true over L.

Proof

Take any element a of L. Const(a) is true over V because a belongs to some $L_{á}$. But because the property of being constructible is absolute, Const(a) is also true over L. Thus all elements of L are constructible.

Theorem 29

Let T be a transitive class such that the axiom of constructibility is true over T. Then every element of T is constructible and its order is in T.

Proof

Assume the hypothesis. Then (Ãx)Const(x) is true over T. Thus every element of T is constructible. So let a be any element of T. Then because (Ãx)Const(x), (y å T)L(a,y) is true over T. Let b be the element of T such that L(a,b) is true over T. We know that L(a,b) is absolute upwards by theorem 24, so L(a,b) is also true over V. This means that b is really an ordinal a å L_b. Thus the order of a $\leq$ b.

Now, since b is an element of T, and T is transitive, all the elements of b are also in T. But b is an ordinal, so this implies that all ordinals less than b are also elements of T. We know that the order of a is an ordinal which is less than or equal to b so the order of a is also an element of T.

Approaching GCH

We are finally prepared to approach our desired theorem, that is, we are to show that the Generalized Continuum Hypothesis is consistent with the axioms of set theory. For now we will accept without proof that **L is a first order universe** where we define a first order universe to be a class over which all the axioms of ZF are true as well as the axiom of well-foundedness (every non empty member of L contains an initial element). Consequently:

a) L is extensional.
b) L is well founded.

These few final steps are undoubtedly the most exciting, for they tie in all that has been covered along the way. Our first task is to show that if the Axiom of Constructibility is true, so is the Generalized Continuum Hypothesis. From this we will show that Generalized Continuum Hypothesis can never be disproved in ZF, unless of course, ZF is inconsistent.

We shall notate the successor of a cardinal c by c*.

Theorem 30 (Cantor's Theorem)

A < P(A) for any set A. That is, A can be put into a 1-1 correspondence with a subset of P(A) but not with P(A).

Proof

Assume that there exists a 1-1 correspondence between A and P(A). Now let us construct the following set S where S is the set of all elements x of A such that x does not belong to S_x (the set corresponding to x in the 1-1 correspondence). Then for every element x of A, S does not equal S_x. The reason is as follows: Say that for y, $S=S_y$. Now consider the element y. y is either in S or it is not in S. If y is in S, than y does not belong to S_y and hence S does not equal S_y. If y is not in S, then y does belong to S_y so again S does not equal S_y. Thus, we have constructed a subset of A, S, which cannot correspond to any member of A. But by our assumption, there is a 1-1 correspondence, so S should correspond to some element of A. Contradiction. Therefore there does not exist a 1-1 correspondence between A and P(A). Now it is easy to show that A can be put into a 1-1 correspondence with a subset of P(A). Simply match each element x of A to {x} in P(A).

The Generalized Continuum Hypothesis is, in essence, saying that card(P(c)) = c* where c* is the least cardinal greater than cardinality of c. We know by Cantor's theorem that the cardinality of P(c) is at least c* (since it is greater than c and c* is the least cardinal greater than c). So, to prove GCH it is sufficient to show that $P(c) \leq c^*$. What we will do now is show that there are at most c* CONSTRUCTIBLE subsets of c. Thus, all that GCH requires is the axiom of constructibility which is consistent with ZF (since it is true over L and L is a first order universe).

Theorem 31

$\mathrm{card}(L_{á}) = \mathrm{card}(á)$

Proof

1) First we must show that an infinite set A of cardinality c is of the same cardinality as the set of finite sequences of elements of A. We know that the set of all the sequences of length 1 is of the same cardinality as c. We also know that 2-sequences are of the same cardinality as c, since in general: A X A is of the same cardinality as A. Sequences of length 3 are also the same size as A since (A X A) X A is of cardinality c as well. By a simple inductive argument, it is obvious that that the cardinality of A is equivalent to the cardinality of finite sequences of elements in A.
2) Since the codes of formulas with constants in A are simply sequences with elements of A, we can conclude that the set of formulas with constants in A has the same cardinality as A.
3) Any formula in A defines some subset of A. So we can map the set of definable subsets of A onto the formulas with constants in A. Thus *F*(A) is the same cardinality as A. Also since $L_{á+1}= F(L_{á})$, by taking $L_{á}$ for A, we can conclude that $L_{á+1}$ has the same cardinality as $L_{á}$.
4) We will now use transfinite induction to show that $L_{á} \leq á$
   Assume that for all $â \leq á$: $L_{á} \leq á$.
   We know that $\mathrm{card}(L_{á+1}) = \mathrm{card}(L_{á})) \leq á$ (by our inductive hypothesis) $\leq á+1$.
   Thus $L_{á+1} \leq á+1$.
   Now, assume that for all $á < ë$ (where ë is a limit ordinal) $L_{á} \leq á$.
   It is tempting to jump to the conclusion that $U_{á<ë}\, L_{á} \leq U_{á<ë}\, á$. However, this is not obvious. Take for example the following:
   A = {1,2,3}
   B = {1}
   C = {1,2,3}
   D = (2,3,4}

Now it is true that B≤A and that D≤C. But consider the union of A,C and the union of B,D. $U_{A,C}$ = {1,2,3}. $U_{B,D}$ = {1,2,3,4}. Thus $U_{B,D} > U_{A,C}$ even though the individual sets B and D are both less than or equal to the individual sets A and C respectively. So how do we resolve this dilemma when trying to derive from the fact that for all ordinal less than ë, $L_{á} \leq$ á, that $L_{ë} \leq$ ë? Well, the problem we encountered before aroused because the elements of C were identical to those of A whereas the elements of B and D had no elements in common. However a special feature of both the $L_{á}$ s and the ordinals is that they are *nested* within one another. So instead consider the following situation which is more comparable to the one we are discussing:

A = {1,2,3}
B = {a,b}
C = {1,2,3,4}
D = {a,b,c}

So for each $L_{â}$ where â is an ordinal less than ë, not only do we know that $L_{â} \leq$ â, we also know that the union of $L_{á}$ where á<â is less than the union of á s where á≤â. This is because the union of these sets is in fact $L_{â}$, just like the union of A and C = C because A is nested within C. Thus, applying this to an infinite such nesting we get that $L_{ë} \leq$ ë.

5) Finally we need to show that for each infinite cardinal c, L_c has at least c elements. We know that $L_{á+1}$ contains all the elements of $L_{á}$ but it also contains $L_{á}$ itself. So $L_{á+1}$ contains at least one more element than $L_{á}$. We also know that $L_{á}$ is of order á, so á is a member of $L_{á}$. But $L_{á}$ is transitive, so all the elements of á are members of $L_{á}$, as well. Thus á is a subset of $L_{á}$ and thus has cardinality less than or equal to $L_{á}$.
6) Combining steps 4 and 5, we get that $L_{á}$ is the same cardinality as á.

Theorem 32 (G)

For any infinite cardinal c, every constructible subset of L_c is an element of L_{c^*}.

Proof

Let m be any constructible subset of L_c. Then, because L_c has cardinality c and m is a subset of L_c, L_c U {m} also has cardinality c. Now, we assumed that L is an extensional well founded class. Also, L can be well ordered (by the ordering of the ordinals which are the indices of the L's) and we proved that the sentence ÃxConst(x) is true over L. Then by the M.S.T.V theorem, m is a subset of some T where T is a transitive class such that ÃxConst(x) is true over T, and T is also of cardinality c. Then since m is a subset of T and T is transitive, m is also an element of T. Also, by theorem 29 m is constructible and the order of m is in T. Let the order of m be á. So á is also a member of T. But since T is transitive, á is then also a subset of T. So the cardinality of á is less than or equal to the cardinality of T which equals c. So á<c* and since á is the order of m, we can conclude that m is an element of L_{c^*}.

Theorem 33

V=L implies the Generalized Continuum Hypothesis.

Proof

By Cantor's theorem we know that P(c)>c.

So $P(c) \geq c^*$.
To show that $P(c) = c^*$ it remains to show that $P(c) \leq c^*$. Every constructible subset of L_c is a member of L_{c^*} by the previous theorem. This is equivalent to saying that the order of every constructible subset of L_c is less than c*, since order of x is defined to be the first á such that x å $L_{á+1}$. But if the order is less than c*, we can say more generally that the order of every constructible subset of L_c is less than or equal to c. We proved in theorem 31 that the cardinality of L_{c^*} is c*. And since every constructible subset of L_c is a member of L_{c^*}, there are at most c* constructible subsets of L_c. But L_c has cardinality c, so it is also true that there at most c* constructible subsets of c. But if, V=L, then every set is constructible and we can say instead that there are at most c* subsets of c. Or in other words, the desired result that $P(c)<c^*$.
Combining the two inequalities we get the famous Generalized Continuum Hypothesis: card(P(c)) = card(c*).

We have just shown that the statement “every set x is constructible” implies the Generalized Continuum Hypothesis is a theorem of the Zermelo Frankel axioms. In particular, L is a first order universe and thus “every set x is constructible” implies GCH is true over L. But we have already shown that “every set x is constructible” is true over L. So

Theorem 34
GCH is true over L.

Theorem 35
GCH is consistent with ZF
Proof
We will give a proof by contradiction.
Say it were true that –GCH (the negation of GCH) were provable in ZF. Then –GCH would be true over all first order universes. In particular, -GCH would be true over L. But by theorem 34, GCH is true over L. So GCH and –GCH is true over L. Contradiction. Therefore, one cannot prove –GCH in ZF and GCH is consistent with ZF.