The Largest Eigenvalue Distribution in the Rank 1 Quaternionic Spiked Model

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Abstract

The spiked models are important examples of the Wishart ensembles in statistics and a generalizations of Laguerre ensembles in random matrix theory. In this paper we find the limiting edge distributions and observe a phase transition phenomenon on the quaternionic field.

1. Introduction

Under the statistical disguise, the rank 1 quaternionic spiked model is that, given an $N \times M$ matrix X whose entries are random quaternions in normal distribution, with $\mu = 0$ and $\sigma = 1 + a$ for those in the last row and $\mu = 0$ and $\sigma = 1$ for others, we define $S = \frac{1}{n} X X^*$ and study the distribution of $\{\lambda_j\}_{j=1,\dots,N}$, the eigenvalues of S. In this paper, we find the limiting distribution of $\max(\lambda_j)$, as $M \to \infty$, $N \to \infty$, and $M/N \to \gamma^2 \geq 1$.

Remark. If $a = 0$, this ensemble is the LSE, and the limit distribution is the "soft edge". And if we change the quaternionic field into the complex one, the problem is solved in [1].

Our result is:

Theorem. *In the rank* 1 *quaternionic spiked model stated above, we have*

If $-1 < a < \gamma^{-1}$,

$$\mathbb{P}\left(\left(\max(\lambda_j) - (1 + \gamma^{-1})^2 \right) \cdot \frac{\gamma (2M)^{2/3}}{(1+\gamma)^{4/3}} \leq T \right) \to F_{\mathrm{GSE}}(T)$$

$$= \left[\det \left(I - \begin{pmatrix} \widehat{S}_4(\xi,\eta) & \widehat{SD}_4(\xi,\eta) \\ \widehat{IS}_4(\xi,\eta) & \widehat{S}_4(\eta,\xi) \end{pmatrix} \chi_{(T,\infty)}(\eta) \right) \right]^{1/2},$$

where

$$\widehat{S}_4(\xi,\eta) = \frac{1}{2} K_{\mathrm{Airy}}(\xi,\eta) - \frac{1}{4} \mathrm{Ai}(\xi) \int_\eta^\infty \mathrm{Ai}(t) dt,$$

$$\widehat{SD}_4(\xi,\eta) = -\frac{\partial}{\partial \eta} \widehat{S}_4(\xi,\eta),$$

$$\widehat{IS}_4(\xi,\eta) = -\int_\xi^\infty \widehat{S}_4(t,\eta) dt.$$

If $a = \gamma^{-1}$,

$$\mathbb{P}\left(\left(\max(\lambda_j) - (1 + \gamma^{-1})^2 \right) \cdot \frac{\gamma (2M)^{2/3}}{(1+\gamma)^{4/3}} \leq T \right) \to F_{\mathrm{GSE1}}(T)$$

$$= \left[\det \left(I - \begin{pmatrix} \overline{\overline{S}}_4(\xi,\eta) & \overline{\overline{SD}}_4(\xi,\eta) \\ \overline{\overline{IS}}_4(\xi,\eta) & \overline{\overline{S}}_4(\eta,\xi) \end{pmatrix} \chi_{(T,\infty)}(\eta) \right) \right]^{1/2},$$

where

$$\overline{\overline{S}}_4(\xi,\eta) = S_4(\xi,\eta) + \frac{1}{2}\mathrm{Ai}(\xi),$$

$$\overline{\overline{SD}}_4(\xi,\eta) = SD_4(\xi,\eta),$$

$$\overline{\overline{IS}}_4(\xi,\eta) = IS_4(\xi,\eta) - \frac{1}{2} \int_\xi^\infty \mathrm{Ai}(t) dt.$$

If $a > \gamma^{-1}$,

$$\mathbb{P}\left(\left(\max(\lambda_j) - (a+1)\left(1 + \frac{1}{\gamma^2 a}\right) \right) \cdot \frac{\sqrt{2M}}{(a+1)\sqrt{1 - \frac{1}{\gamma^2 a^2}}} \leq T \right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^T e^{-\frac{t^2}{2}} dt.$$

2. Quaternionic Zonal Polynomials

First, the joint distribution of $\{\lambda_j\}$ is

$$P(\lambda) = \frac{1}{C} (V(\lambda))^4 \prod_{j=1}^N \lambda_j^{2(M-N)+1} e^{-2M\lambda_j} \int_{Q \in Sp(N)} e^{2M \mathrm{Tr}(AQ\Lambda Q^{-1})} dQ,$$

with $A = \mathrm{diag}(\frac{a}{1+a}, 0, \dots, 0)$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$.

Then we expand the integral by quaternionic Zonal polynomials [3]

$$\int_{Q \in Sp(N)} e^{2M \mathrm{Tr}(AQ\Lambda Q^{-1})} dQ = \sum_{j=0}^\infty \frac{(2M)^j}{j!} \sum_{\substack{l(\kappa) \leq N \\ \kappa \vdash j}} \frac{C_\kappa^{(1/2)}(A) C_\kappa^{(1/2)}(\Lambda)}{C_\kappa^{(1/2)}(I_N)}, \quad (1)$$

where $C_\kappa^{(1/2)}$ is the quaternionic Zonal polynomial, i.e., the Jack polynomial with the parameter $\alpha = 1/2$ and the C-normalization, so that

$$\sum_{\substack{l(\kappa) \leq m \\ \kappa \vdash k}} C_\kappa^{(1/2)}(x_1, \dots, x_m) = (x_1 + \dots + x_m)^k.$$

Since we have [3]

$$C_\kappa^{(1/2)}(A) = C_\kappa^{(1/2)}(\frac{a}{1+a}, 0, \dots, 0) = \begin{cases} \left(\frac{a}{1+a}\right)^j & \kappa = (j), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C_{(j)}^{(1/2)}(I_N) = \frac{1}{(j+1)!} \prod_{i=0}^{j-1} (2N + i),$$

we simplify (1) as

$$\int_{Q \in Sp(N)} e^{2M \mathrm{Tr}(AQ\Lambda Q^{-1})} dQ = \sum_{j=0}^\infty \frac{j+1}{\prod_{i=0}^{j-1}(2N+i)} \left(\frac{a}{1+a} 2M \right)^j C_{(j)}^{(1/2)}(\Lambda). \quad (2)$$

From [3]

$$\sum_{j=0}^\infty (j+1) C_{(j)}^{(1/2)}(\Lambda) t^j = \prod_{j=1}^N \frac{1}{(1 + \lambda_j t)^2},$$

and comparing with the well known formula for Schur polynomials

$$\sum_{j=0}^\infty s_{(j)}(\Lambda) t^j = \prod_{j=1}^N \frac{1}{1 + \lambda_j t},$$

we get the identity

$$(j+1) C_{(j)}^{(1/2)}(\Lambda) = s_{(j)}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_N, \lambda_N), \quad (3)$$

with each λ_j repeating twice as variables of $s_{(j)}$.

Next, we apply the classical determinantal representation of Schur polynomials to the degenerate case $c_{2j-1} = c_{2j} = \lambda_j$. With L'Hôpital's rule, we get

$$s_{(j)}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_N, \lambda_N) = \frac{\begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & \frac{d}{d\lambda_1}\lambda_1^{2N-2} & \dots & \lambda_N^{2N-2} & \frac{d}{d\lambda_N}\lambda_N^{2N-2} \\ \lambda_1^{2N+j-1} & \frac{d}{d\lambda_1}\lambda_1^{2N+j-1} & \dots & \lambda_N^{2N+j-1} & \frac{d}{d\lambda_N}\lambda_N^{2N+j-1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-1} & \frac{d}{d\lambda_1}\lambda_1^{2N-1} & \dots & \lambda_N^{2N-1} & \frac{d}{d\lambda_N}\lambda_N^{2N-1} \end{vmatrix}}. \quad (4)$$

Here we notice the denominator in (4) is equal to $V(\lambda)^4$.

Now substituting (3) and (4) into (2), we get

$$P(\lambda) = \frac{1}{C} \tilde{V}^4(\lambda) \prod_{j=1}^N \left(\lambda_j^{2(M-N)+1} e^{-2M\lambda_j} \right),$$

where

$$\tilde{V}^4(\lambda) = \begin{vmatrix} 1 & 0 & \dots & 1 & 0 \\ \lambda_1 & 1 & \dots & \lambda_N & 1 \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_N^2 & 2\lambda_N \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \lambda_1^{2N-2} & \frac{d}{d\lambda_1}\lambda_1^{2N-2} & \dots & \lambda_N^{2N-2} & \frac{d}{d\lambda_N}\lambda_N^{2N-2} \\ e^{\frac{a}{1+a}2M\lambda_1} & \frac{d}{d\lambda_1} e^{\frac{a}{1+a}2M\lambda_1} & \dots & e^{\frac{a}{1+a}2M\lambda_N} & \frac{d}{d\lambda_N} e^{\frac{a}{1+a}2M\lambda_N} \end{vmatrix}$$

3. Skew Orthogonal Polynomials

If we take an anti-symmetric product

$$\langle f(x), g(x) \rangle_4 = \int_0^\infty (f(x)g'(x) - f'(x)g(x)) x^{2(M-N)+1} e^{-2Mx} dx,$$

and choose skew orthogonal functions (mostly polynomials except for the last) $\varphi_0, \varphi_1, \dots, \varphi_{2N-1}$ such that for $j = 0, \dots, 2N-2$, $\varphi_j(x)$ is a linear combination of $\{1, x, \dots, x^j\}$ while φ_{2N-1} is a linear combination of $\{1, \dots, x^{2N-2}, e^{\frac{a}{1+a}2M}\}$, and

$$\langle \varphi_j(x), \varphi_k(x) \rangle_4 = \begin{cases} r_i & \text{if } j = 2i \text{ and } k = 2i+1, \\ -r_i & \text{if } j = 2i+1 \text{ and } k = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Then denoting $\psi_j(x) = \varphi_i(x) x^{M-N+1/2} e^{-Mx}$, we get by an argument in [4],

$$\mathbb{P}^2(\max(\lambda_j) \leq T) = \left[\int_0^T \dots \int_0^T P(\lambda) d\lambda_1 \dots \lambda_N \right]^2 = \det \left(I - K_4(x,y) \chi_{(T,\infty)}(y) \right),$$

where

$$K_4(x,y) = \begin{pmatrix} S_4(x,y) & SD_4(x,y) \\ IS_4(x,y) & S_4(y,x) \end{pmatrix},$$

with

$$S_4(x,y) = \sum_{j=0}^{N-1} \frac{1}{r_j} (-\psi'_{2j}(x)\psi_{2j+1}(y) + \psi'_{2j+1}(x)\psi_{2j}(y)),$$

and $SD_4(x,y)$ and $IS_4(x,y)$ defined similarly.

4. Asymptotic Analysis

Here we only analyze $S_4(x,y)$ for example.

We split $S_4(x,y)$ as $S'_4(x,y) + S''_4(x,y)$, where

$$S'_4(x,y) = \sum_{j=0}^{N-2} \frac{1}{r_j} (-\psi'_{2j}(x)\psi_{2j+1}(y) + \psi'_{2j+1}(x)\psi_{2j}(y)),$$

$$S''_4(x,y) = \frac{1}{r_{N-1}} (-\psi'_{2N-2}(x)\psi_{2N-1}(y) + \psi'_{2N-1}(x)\psi_{2N-2}(y)).$$

If we change variables and take

$$S_4(\xi,\eta) = \frac{(1+\gamma)^{4/3}}{\gamma M^{2/3}} S_4(x,y) \Big|_{\substack{x = (1+\gamma^{-1})^2 + \frac{(1+\gamma)^{4/3}}{\gamma M^{2/3}}\xi \\ y = (1+\gamma^{-1})^2 + \frac{(1+\gamma)^{4/3}}{\gamma M^{2/3}}\eta}},$$

then we get [2]

$$S'_4(\xi,\eta) \to \widehat{S}(\xi,\eta),$$

and

- if $a < \gamma^{-1}$,

$$S''_4(\xi,\eta) \to 0;$$

- if $a = \gamma^{-1}$, upon conjugation,

$$S''_4(\xi,\eta) \to \frac{1}{2}\mathrm{Ai}(\xi).$$

For $a > \gamma^{-1}$, we change variables an take

$$S_4(\xi,\eta) = \frac{(1+a)\sqrt{1 - \frac{1}{\gamma^2 a^2}}}{\sqrt{2M}} S_4(x,y) \Big|_{\substack{x = (a+1)(1+\frac{1}{\gamma^2 a}) + \frac{(1+a)\sqrt{1-\frac{1}{\gamma^2 a^2}}}{\sqrt{2M}}\xi \\ y = (a+1)(1+\frac{1}{\gamma^2 a}) + \frac{(1+a)\sqrt{1-\frac{1}{\gamma^2 a^2}}}{\sqrt{2M}}\eta}},$$

then

$$S'_4(\xi,\eta) \to 0$$

and upon conjugation,

$$S''_4(\xi,\eta) \to s(\xi,\eta) + s(\eta,\xi),$$

where

$$s(\xi,\eta) = \frac{1}{2\sqrt{2\pi}} e^{\frac{(\gamma^2 a^2 - 1)M}{(\gamma^2 a + 1)(a+1)}(y-x)} e^{-\frac{\gamma^4 a^2 + \gamma^2 a^2 + 4\gamma^2 a + \gamma^2 + 1}{4(\gamma^2 a - 1)^2}\xi^2 - \frac{(\gamma^2 a^2 - 1)(\gamma^2 - 1)}{4(\gamma^2 a - 1)^2}\eta^2}.$$

After a conjugation, we get

$$K_4(\xi,\eta) \to \begin{pmatrix} s(\xi,\eta) + s(\eta,\xi) & -s(\xi,\eta) + s(\eta,\xi) \\ -s(\xi,\eta) + s(\eta,\xi) & s(\xi,\eta) + s(\eta,\xi) \end{pmatrix},$$

and finally

$$\det\left(I - K_4(\xi,\eta)\chi_{(T,\infty)}(\eta) \right) \to \det\left(I - \begin{pmatrix} 2s(\xi,\eta) & 0 \\ 0 & s(\eta,\xi) \end{pmatrix} \chi_{(T,\infty)}(\eta) \right) = \left[\frac{1}{\sqrt{2\pi}} \int_\infty^T e^{-\frac{t^2}{2}} dt \right]^2.$$

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