

NOTES ON OVERLAP ESTIMATES

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1. THE GROUP RING OF \mathbb{Q}/\mathbb{Z}

First we recall some relevant results about the group ring of \mathbb{Q}/\mathbb{Z} . Let G be the multiplicative group defined by

$$G = \{z^\beta : \beta \in \mathbb{Q}/\mathbb{Z}\},$$

and let $(\mathbb{Z}(G), +, \times)$ denote the group ring of G with coefficients in \mathbb{Z} . We are not hiding the fact that G is just \mathbb{Q}/\mathbb{Z} written multiplicatively, but depending on the context it may be more appropriate to speak either of addition in \mathbb{Q}/\mathbb{Z} or of multiplication in G . Of course the group ring $\mathbb{Z}(G)$ is just the ring of multivariable polynomials with integer coefficients, evaluated at elements of G .

For each $\beta \in \mathbb{Q}/\mathbb{Z}$ we denote the additive order of β by $h(\beta)$. Then for each positive integer Q we define a subset \mathcal{F}_Q of \mathbb{Q}/\mathbb{Z} by

$$\mathcal{F}_Q = \{\beta \in \mathbb{Q}/\mathbb{Z} : h(\beta) \leq Q\}.$$

For $Q \geq 3$ the set

$$\{z^\beta \in G : \beta \in \mathcal{F}_Q\} \subset G$$

is not closed under multiplication. However it does generate a finite subgroup of G , which we call

$$G_Q = \langle z^\beta : \beta \in \mathcal{F}_Q \rangle.$$

Obviously this definition of G_Q is also valid if $Q < 3$. It is easy to check that

$$(1) \quad G_Q = \{z^\beta : \beta \in \mathbb{Q}/\mathbb{Z}, h(\beta) \mid \text{lcm}\{1, 2, \dots, Q\}\}.$$

The cardinality of G_Q is given by

$$|G_Q| = \sum_{q \mid \text{lcm}\{1, 2, \dots, Q\}} \varphi(q) = \text{lcm}\{1, 2, \dots, Q\} = \exp\left(\sum_{q \leq Q} \Lambda(q)\right),$$

where Λ denotes the von-Mangoldt function. Thus by the Prime Number Theorem there is a positive constant c_1 for which

$$(2) \quad |G_Q| = \exp\left(Q + O\left(Qe^{-c_1\sqrt{\log Q}}\right)\right).$$

It appears to be a somewhat more difficult problem to estimate the cardinality of the set

$$(3) \quad \overbrace{\mathcal{F}_Q + \dots + \mathcal{F}_Q}^{k\text{-times}}$$

when $k \geq 2$ is a small positive integer. To investigate this problem and more generally to count multiplicities in the resulting sets we define for each nonnegative integer q an element $F_q \in \mathbb{Z}(G)$ by

$$F_q = \sum_{\substack{\beta \in \mathbb{Q}/\mathbb{Z} \\ h(\beta)=q}} z^\beta = \sum_{\substack{a=1 \\ (a,q)=1}}^q z^{a/q}.$$

For our overlap estimates we will use the following result, which is proved elsewhere.

Theorem 1. *Suppose that q and r are positive integers. Let $d = (q, r)$ and let d' be the largest divisor of d which is relatively prime to both q/d and r/d . Then we have that*

$$(4) \quad F_q \times F_r = \varphi(d) \sum_{e|d'} c(d', e) F_{qr/de},$$

where

$$c(d', e) = \prod_{\substack{p|d' \\ p \nmid e}} \left(1 - \frac{1}{p-1}\right).$$

2. OVERLAP ESTIMATES

In this section we will use the following standard notation from elementary number theory: p denotes a prime number, $\mu(n)$ is the Möbius function, $\varphi(n)$ is the Euler phi function, $\omega(n)$ denotes the number of distinct prime divisors of n , and $\sigma_0(n)$ is the number of positive integers which divide n . Also we use λ to denote Lebesgue measure on \mathbb{R}/\mathbb{Z} and for $x \in \mathbb{R}/\mathbb{Z}$ we write $\|x\|_{\mathbb{Z}}$ to denote the distance from x to the nearest integer.

Let $\Psi : \mathbb{N} \rightarrow \mathbb{R}$ be a nonnegative arithmetical function and for each positive integer q define $\mathcal{E}_q \subseteq [0, 1]$ by

$$\mathcal{E}_q = \bigcup_{\substack{a=1 \\ (a,q)=1}}^q \left(\frac{a - \Psi(q)}{q}, \frac{a + \Psi(q)}{q} \right).$$

An important problem in metric number theory is to obtain suitable estimates for the measures of the ‘overlaps’ $\mathcal{E}_q \cap \mathcal{E}_r$ for $q \neq r$. Following the notation in the literature, for $q \neq r$ we define the quantities

$$\delta(q, r) = 2 \min \left\{ \frac{\Psi(q)}{q}, \frac{\Psi(r)}{r} \right\}, \quad \text{and}$$

$$\Delta(q, r) = 2 \max \left\{ \frac{\Psi(q)}{q}, \frac{\Psi(r)}{r} \right\}.$$

We will use Theorem 1 together with the Large Sieve Inequality to prove the following lemma.

Lemma 1. *Suppose that $q \neq r$ and let d and d' be as in the statement of Theorem 1. Then for every $Q \geq 1$*

$$\lambda(\mathcal{E}_q \cap \mathcal{E}_r) \leq 2\delta(q, r)\varphi(d) \left(\frac{qr\Delta(q, r)}{d} + Q^2 \right) \frac{1}{G(Q, q, r)},$$

where

$$G(Q, q, r) = \sum_{\substack{e|qr/dd' \\ e \leq Q}} \frac{\mu(e)^2}{\varphi(e)}.$$

This lemma appears to be comparable to the best overlap estimates. The ‘trivial estimate’ obtained by setting $Q = 1$ gives

$$\lambda(\mathcal{E}_q \cap \mathcal{E}_r) \leq 4\delta(q, r)\varphi(d) \frac{qr\Delta(q, r)}{d} \leq 4\Psi(q)\Psi(r),$$

which is precisely what is needed to prove the Duffin-Schaeffer Theorem. On the other hand choosing $Q^2 \approx qr\Delta(q, r)/d$ produces something which is comparable to the Pollington and Vaughan upper bound. Since Q is allowed to be any positive integer, perhaps there is room for some improvement here.

Proof of Lemma 1. We begin with the classical observation that

$$\lambda(\mathcal{E}_q \cap \mathcal{E}_r) \leq \delta(q, r)I(q, r),$$

where $I(q, r)$ denotes the cardinality of the set of pairs of positive integers (a, b) which satisfy $a \leq q$, $b \leq r$, $(a, q) = (b, r) = 1$, and

$$(5) \quad \left| \frac{a}{q} - \frac{b}{r} \right| < \Delta(q, r).$$

Equation (5) is satisfied if and only if

$$|ar - bq| < qr\Delta(q, r)$$

and this can clearly not happen unless $d < qr\Delta(q, r)$. We point this out only for reference, as this hypothesis is not necessary for our proof. Next observe that

$$I(q, r) = \#\{\gamma \in F_q \times F_r \mid \|\gamma\|_{\mathbb{Z}} \leq \Delta(q, r)\}.$$

where the number of elements in the set on the right is counted with multiplicity. By Theorem 1 we then have that

$$\begin{aligned} I(q, r) &= \varphi(d) \sum_{e|d'} c(d', e) \#\{\gamma \in F_{qr/de} \mid \|\gamma\|_{\mathbb{Z}} \leq \Delta(q, r)\} \\ &\leq 2\varphi(d) \sum_{e|d'} \#\{\gamma \in \mathcal{F} \mid h(\gamma) = qr/de, \gamma \leq \Delta(q, r)\} \\ &= 2\varphi(d) \sum_{e|d'} \#\{a \in \mathbb{N} \mid (a, qr/de) = 1, a \leq qr\Delta(q, r)/de\} \\ &= 2\varphi(d) \#\{a \in \mathbb{N} \mid (a, qr/dd') = 1, a \leq qr\Delta(q, r)/d\}. \end{aligned}$$

Now we use the Large Sieve to estimate the cardinality of the set of integers appearing here. Let

$$\begin{aligned} \mathcal{P} &= \{p \mid qr/dd' \mid p \text{ prime}\}, \\ \omega(p) &= \begin{cases} 1 & \text{if } p \in \mathcal{P} \\ 0 & \text{else} \end{cases}, \text{ and} \\ h(a) &= \mu(a)^2 \prod_{p|a} \frac{\omega(p)}{p - \omega(p)}. \end{aligned}$$

Then

$$\sum_{a \leq Q} h(a) = G(Q, q, r),$$

and the Large Sieve Inequality gives us

$$\#\{a \in \mathbb{N} \mid (a, qr/dd') = 1, a \leq qr\Delta(q, r)/d\} \leq \frac{\frac{qr\Delta(q, r)}{d} + Q^2}{G(Q, q, r)}$$

for all $Q \geq 1$. This completes the proof of the lemma. \square