A GENERALIZATION OF THE FAREY INDEX

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Abstract. We consider a generalization $\nu_k$ of the Farey index $\nu$ defined by R. R. Hall and P. Shiu. Given a positive integer $Q$ the values of $\nu_k$ in our definition are obtained by taking the numerators of differences between Farey fractions which are separated by $k-1$ elements in the Farey sequence of order $Q$. We prove that the values of $\nu_k$ are given by the $(k-1)$st convergent polynomial, evaluated at various values of $\pm \nu_2$. This fact allows us employ methods used by F. P. Boca, R. N. Gologan, and A. Zaharescu in order to prove an asymptotic formula for the sum over all Farey points of order $Q$ of the values of $\nu_k$.

1. Introduction

Let $F_Q = \{\gamma_1, \gamma_2, \ldots, \gamma_{N(Q)}\}$ denote the Farey sequence of order $Q$, arranged so that $1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N(Q)} = 1$. We will extend the sequence $F_Q$ by requiring that $\gamma_{i+N(Q)} = \gamma_i + 1$ for all $i \in \mathbb{Z}$. For each $i$ we then write $\gamma_i = p_i/q_i$ with gcd($p_i, q_i$) = 1. The index of the fraction $\gamma_i$ was defined in [4] as

$$\nu(\gamma_i) = \frac{q_{i-1} + q_{i+1}}{q_i}.$$  

This function has been the object of study of several authors. Hall and Shiu [4] proved that

$$\sum_{i=1}^{N(Q)} \nu(\gamma_i) = 3N(Q) - 1.$$  

In the same paper they also proved asymptotic formulas for the sum of the square moments of $\nu$, as well as for several other related quantities. Boca, Gologan, and Zaharescu [2] extended this result by finding asymptotic formulas for all moments of the index which lie in $(0, 2)$. Among other things they also proved that for $h \geq 1$ there exists a constant $A(h)$ for which

$$\sum_{i=1}^{N(Q)} \nu(\gamma_i)\nu(\gamma_{i+h}) = A(h)N(Q) + O_h(Q \log^2 Q).$$  

In [5] Hall has also studied the restriction of the Farey index to fractions with odd or even denominators.

In this paper we will consider a generalization of the Farey index which turns out to satisfy some attractive identities. The definition (1) can be written in two
alternate forms. First of all, as recorded in [4, (1.4)], we have that
\[ \nu(\gamma_i) = \left[ \frac{Q + q_i - 1}{q_i} \right]. \]

Secondly by the basic properties of Farey fractions we find that
\[ \nu(\gamma_i) = \frac{q_i - 1 + q_{i+1}}{q_i} = q_i + 1 q_i - 1 \left( \frac{1}{q_i - 1 q_i} + \frac{1}{q_q i_i + 1} \right) \]
\[ = q_i - 1 q_i + 1 (\gamma_{i+1} - \gamma_i + \gamma_i - \gamma_i - 1) \]
\[ = p_i + 1 q_i - 1 - p_i - 1 q_i + 1. \]

This second observation is the basis for the following definition.

**Definition.** Given a positive integer \( k \) and a fraction \( \gamma_i \) in \( \mathcal{F}_Q \) we define the \( k \)-index of \( \gamma_i \) by
\[ \nu_k(\gamma_i) = p_i + k q_i - 1 - p_i - 1 q_i + k - 1. \]

Several remarks are in order. First note that our definition of \( \nu_k \) depends on \( Q \). Second by (5) we see that the index \( \nu \) is recovered as the \( 2 \)-index \( \nu_2 \). Also by the determinant property of Farey fractions the \( 1 \)-index \( \nu_1 \) takes the constant value 1.

Finally the functions \( \nu_k \) are easily seen to be periodic in the sense that
\[ \nu_k(\gamma_{i+N(Q)}) = p_{i+k-1} q_{i+1} + N(Q) - p_i - 1 q_i - (p_i + q_i - 1) q_i + k - 1 \]
\[ = p_i + k q_i - 1 - p_i - 1 q_i + k - 1 \]
\[ = \nu_k(\gamma_i). \]

Here we are using the facts that for all \( i \)
\[ p_{i+N(Q)} = p_i + q_i \quad \text{and} \quad q_{i+N(Q)} = q_i. \]

We now state the main results contained in this paper. In Section 2 we will demonstrate a functional relationship between the \( k \)-indices and the convergent polynomials \( \{K_n\}_{n=0} \). The convergent polynomials (see [3]) are defined by
\[ K_0(x) = 1 \quad \text{and} \quad K_1(x) = x, \]
and then recursively by
\[ K_n(x_1, x_2, \ldots, x_n) = x_n K_{n-1}(x_1, x_2, \ldots, x_{n-1}) + K_{n-2}(x_1, x_2, \ldots, x_{n-2}). \]

Thus the polynomial \( K_n \) is an element of \( \mathbb{Z}[x_1, x_2, \ldots, x_n] \) and is linear in each of the variables \( x_1, x_2, \ldots, x_n \). For example we have
\[ K_2(x_1, x_2) = x_1 x_2 + 1 \quad \text{and} \quad K_3(x_1, x_2, x_3) = x_1 x_2 x_3 + x_1 + x_3. \]

We will prove the following result.

**Theorem 1.** Let \( Q \) be fixed and let \( k \) be any positive integer. Then for each fraction \( \gamma_i \) in \( \mathcal{F}_Q \) we have
\[ \nu_k(\gamma_i) = \left( \frac{2k - 1}{2} \right) K_{k-1} \left( -\nu_2(\gamma_i), \nu_2(\gamma_{i+1}), \ldots, (-1)^{k-1} \nu_2(\gamma_{i+k-2}) \right), \]
where \( \left( \frac{n}{2} \right) \) denotes the Kronecker symbol, defined by

\[
\left( \frac{n}{2} \right) = \begin{cases} 
0 & \text{if } 2 \nmid n, \\
1 & \text{if } n \equiv \pm 1 \mod 8, \text{ and} \\
-1 & \text{if } n \equiv \pm 3 \mod 8.
\end{cases}
\]

In Section 3 we will show how Theorem 1 can be used to prove the following asymptotic estimates.

**Theorem 2.** For each integer \( k \geq 0 \) there exists a real constant \( B(k) \) for which

\[
\sum_{i=1}^{N(Q)} \nu_k(\gamma_i) = B(k)N(Q) + O_k(Q \log^2 Q),
\]

as \( Q \to \infty \).

Our proof of this fact relies on an extension of the methods used to establish (3). As the proof will show, for each \( k \) the constant \( B(k) \) in the statement of the theorem can be determined by the area of a subregion of the Farey triangle (defined below).

2. **Identities and Formulas**

In this section we will establish some basic identities satisfied by the \( k \)–indices and we will provide a proof of Theorem 1. First we prove the following fundamental identity.

**Lemma 1.** Choose \( Q \geq 2 \) and \( \gamma_i \in F_Q \). Then for \( k \geq 3 \) we have

\[
\begin{pmatrix}
\nu_{k-1}(\gamma_i) \\
\nu_{k-2}(\gamma_{i+1}) \\
\nu_{k-1}(\gamma_{i+1})
\end{pmatrix} \in SL_2(\mathbb{Z}).
\]

**Proof.** We need to show that the determinant of the matrix on the left hand side of (11) is equal to one. First we have

\[
\nu_{k-1}(\gamma_i)\nu_{k-1}(\gamma_{i+1}) = (p_{i+k-k-2}q_{i-1} - p_{i-1}q_{i+k-2})(p_{i+k-k}q_{i+k-1} - p_{i}q_{i+k-1})
\]

\[
= p_{i+k-k}p_{i+k-k-1}q_{i-1} + p_{i-1}p_{i+k-k-2}q_{i+k-1}
\]

\[
- p_{i-1}p_{i+k-k}q_{i+k-2} - p_{i}p_{i+k-k-2}q_{i-1}.
\]

The sum of the first two terms on the right hand side of this equation is

\[
p_{i+k-k}p_{i+k-k-1}q_{i-1} + p_{i-1}p_{i+k-k-2}q_{i+k-1}
\]

\[
= p_{i+k-k}q_{i-1} + p_{i+k-k}q_{i+k-2} + p_{i-1}q_{i+k-2} - p_{i}q_{i+k-j} = 1,
\]

By the determinant property of Farey fractions we also have that

\[
p_{i+k-k}p_{i+k-k-1}q_{i+k-2} + p_{i-1}p_{i+k-k-2}q_{i+k-1}
\]

\[
- p_{i-1}p_{i+k-k}q_{i+k-2} - p_{i}p_{i+k-k-2}q_{i-1} = 1
\]

Combining (12), (13), and (14) we find that

\[
\nu_{k-1}(\gamma_i)\nu_{k-1}(\gamma_{i+1}) = \nu_{k}(\gamma_i)\nu_{k-2}(\gamma_{i+1}) = 1.
\]
Lemma 2. For actually a special type of polynomial. For this we will use the following result.

\[ \nu_k(\gamma_i) = \nu_{k-1}(\gamma_i)\nu_{k-1}(\gamma_{i+1}) - 1 \]

Induction now shows that \( \nu_k(\gamma_i) \) is given by a rational function evaluated at the integers \( \{\nu_2(\gamma_j)\}_{j=i-k+2}^{i+k-2} \). Of course our goal is to prove that this rational function is actually a special type of polynomial. For this we will use the following result.

**Lemma 2.** For \( k \geq 3 \) and \( \gamma_i \in \mathcal{F}_Q \) we have

\[ \nu_k(\gamma_i) = \nu_2(\gamma_{i+k-2})\nu_{k-1}(\gamma_i) - \nu_{k-2}(\gamma_i). \]

**Proof.** Our proof is by induction. Setting \( k = 3 \) in identity (16) and using the fact that \( \nu_1 \equiv 1 \) gives us

\[ \nu_3(\gamma_i) = \frac{\nu_2(\gamma_i)\nu_2(\gamma_{i+1}) - 1}{\nu_1(\gamma_{i+1})} = \nu_2(\gamma_i)\nu_2(\gamma_{i+1}) - \nu_1(\gamma_i). \]

Now assume the truth of our identity for all integers \( 3 \leq j \leq k-1 \). Then using (16) together with the inductive hypothesis we find that

\[ \nu_k(\gamma_i) = \nu_k(\gamma_{i+k-2})\nu_{k-1}(\gamma_i) - \nu_{k-2}(\gamma_i). \]

Now assuming \( k = 3 \) for \( k \) in (15) we find that the numerator of the fraction in (18) is equal to zero, and this finishes the proof.

With Lemma 2 in hand we are now ready to give a proof of Theorem 1.

**Proof of Theorem 1.** Again the proof is by induction on \( k \). For \( k = 1 \) the function on the right hand side of (9) is

\[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} K_0(\cdot) = 1 = \nu_1(\gamma_i), \]

and for \( k = 2 \) it is

\[ \begin{pmatrix} 3 \\ 2 \end{pmatrix} K_1(-\nu_2(\gamma_i)) = \nu_2(\gamma_i). \]

Now assume that \( k \geq 3 \) and that our result is true for all integers \( 1 \leq j \leq k-1 \). Then using (17) gives us

\[ \nu_k(\gamma_i) = \nu_2(\gamma_{i+k-2})\nu_{k-1}(\gamma_i) - \nu_{k-2}(\gamma_i) \]

\[ = \left( \frac{2k-3}{2} \right) \nu_2(\gamma_{i+k-2})K_{k-2}(-\nu_2(\gamma_i), \ldots, (-1)^{k-2}\nu_2(\gamma_{i+k-3})) \]

\[ - \left( \frac{2k-5}{2} \right) K_{k-3}(-\nu_2(\gamma_i), \ldots, (-1)^{k-3}\nu_2(\gamma_{i+k-4})). \]

We also observe by (6) that

\[ \left( \frac{2k-1}{2} \right) K_{k-1}(-\nu_2(\gamma_i), \ldots, (-1)^{k-1}\nu_2(\gamma_{i+k-2})). \]
using these results and the formula (7) we find that with (3) immediately yields a proof of Theorem 2 in the case when $k$.

Finally, combining (19) and (20) establishes (9).

Note that here we have used the properties of the Kronecker symbol to write

$$
\sum_{i=1}^{N(Q)} \left( \frac{2k-1}{2} \right) \left( \frac{4k-3}{2} \right) = \left( \frac{8k^2 - 10k + 3}{2} \right) = \left( \frac{2k-3}{2} \right)
$$

and

$$
\left( \frac{2k-1}{2} \right) = - \left( \frac{2k-5}{2} \right).
$$

Finally, combining (19) and (20) establishes (9).

To conclude this section we would like to point out that Theorem 1 together with (3) immediately yields a proof of Theorem 2 in the case when $k = 3$. Indeed using these results and the formula (7) we find that

$$
\sum_{i=1}^{N(Q)} \nu_2(\gamma_i) = \sum_{i=1}^{N(Q)} \left( \frac{5}{2} \right) K_2(-\nu_2(\gamma_i), \nu_2(\gamma_{i+1}))
$$

$$
= \sum_{i=1}^{N(Q)} (\nu_2(\gamma_i)\nu_2(\gamma_{i+1}) - 1)
$$

$$
= (A(1) - 1)N(Q) + O(Q \log^2 Q),
$$

where $A(1)$ is the constant obtained from (3) with $h = 1$.

3. Asymptotic Results

Now we will show how the results of Section 2 can be used to prove Theorem 2 in its full generality. Following the notation of [1] and [2] we define the Farey triangle $\mathcal{T} \subseteq [0, 1]^2$ by

$$
\mathcal{T} = \{(x, y) \in [0, 1]^2 : x + y > 1\},
$$

and we define the map $T : [0, 1]^2 \to [0, 1]^2$ by

$$
T(x, y) = \left( y, \frac{1+x}{y} y - x \right),
$$

where $[x]$ denotes the greatest integer less than or equal to $x$. As shown in [1], the map $T$ is a one-to-one area preserving transformation of $\mathcal{T}$ onto itself. Now for
each positive integer \( k \) let
\[
T_k = \left\{ (x, y) \in T : \left\lfloor \frac{1 + x}{y} \right\rfloor = k \right\}.
\]

Then the set \( T \) is the disjoint union of the sets \( T_k \) and we also have that
\[
T(x, y) = (y, ky - x) \quad \text{for all} \quad (x, y) \in T_k.
\]

Also of importance to us are the maps \( \kappa_i : T \to \mathbb{Z}^+ \) defined for positive integers \( i \) by
\[
\kappa_1 = \left\lfloor \frac{1 + x}{y} \right\rfloor \quad \text{and} \quad \kappa_{i+1} = \kappa_i \circ T(x, y) = \kappa_1 \circ T^i(x, y).
\]

It is clear from the definition that
\[
(21) \quad T_k = \left\{ (x, y) \in T : \kappa_1(x, y) = k \right\}.
\]

One of the most useful properties of the function \( T \) is the fact that for any integer \( j \) we have
\[
T \left( \frac{q_{j-1}}{Q}, \frac{q_j}{Q} \right) = \left( \frac{q_j}{Q}, \frac{q_{j+1}}{Q} \right).
\]

Using this fact we find that for any non-negative integer \( i \)
\[
(22) \quad \kappa_{i+1} \left( \frac{q_{j-1}}{Q}, \frac{q_j}{Q} \right) = \kappa_1 \circ T^i \left( \frac{q_{j-1}}{Q}, \frac{q_j}{Q} \right)
\]
\[
= \kappa_1 \left( \frac{q_{j+i-1}}{Q}, \frac{q_{j+i}}{Q} \right)
\]
\[
= \left[ \frac{Q + q_{j+i-1}}{q_{j+i}} \right]
\]
\[
= \nu_2(\gamma_{j+i}),
\]

where the last equality come from the identity (4). We wish to make it clear that none of the notation or results presented thus far in this section are new. A complete exposition of all of these results may be found by consulting the appropriate references.

We will demonstrate the proof of Theorem 2 for any \( k \geq 0 \) but before we do so it will be instructive to first consider how our proof of this theorem works when \( k = 4 \). First of all by Theorem 1 together with formula (8) we have that
\[
(23) \quad \sum_{i=1}^{N(Q)} \nu_4(\gamma_i) = \sum_{i=1}^{N(Q)} \left( \frac{7}{2} \right) K_3(-\nu_2(\gamma_i), \nu_2(\gamma_{i+1}), -\nu_2(\gamma_{i+2}))
\]
\[
= \sum_{i=1}^{N(Q)} \nu_2(\gamma_i)\nu_2(\gamma_{i+1})\nu_2(\gamma_{i+2}) - \sum_{i=1}^{N(Q)} \nu_2(\gamma_i) - \sum_{i=1}^{N(Q)} \nu_2(\gamma_{i+2})
\]
\[
= S_1 - S_2 - S_3.
\]

Appealing to (2) and to the periodicity of \( \nu_2 \) we have that
\[
(24) \quad S_2 = S_3 = 3N(Q) - 1.
\]

To evaluate \( S_1 \) we will use the following well known fact about Farey fractions.
Lemma 3. Let \(a, b,\) and \(Q\) be positive integers. Then there is an integer \(1 \leq i \leq N(Q)\) for which \(q_i = a\) and \(q_{i+1} = b\) if and only if
\[
1 \leq a, b \leq Q, \quad (a, b) = 1, \quad \text{and} \quad a + b > Q.
\]
Furthermore when these conditions on \(a\) and \(b\) are satisfied then the integer \(i\) is uniquely determined.

Using (22) together with this lemma we find that
\[
S_1 = \sum_{i=1}^{N(Q)} \kappa_1 \left( \frac{q_i - 1}{Q}, \frac{q_i}{Q} \right) \kappa_2 \left( \frac{q_i - 1}{Q}, \frac{q_i}{Q} \right) \kappa_3 \left( \frac{q_i - 1}{Q}, \frac{q_i}{Q} \right)
\]
\[(25) = \sum_{(a,b) \in QT \cap Z^2_{\text{vis}}} \kappa_1 \left( \frac{a}{Q}, \frac{b}{Q} \right) \kappa_2 \left( \frac{a}{Q}, \frac{b}{Q} \right) \kappa_3 \left( \frac{a}{Q}, \frac{b}{Q} \right),\]

where here we are using the notation
\[
Z_{\text{vis}}^2 = \{(a,b) \in \mathbb{Z}^2 : (a,b) = 1\}.
\]

Now using (21) and the fact that \(QT \cap Z_{\text{vis}}^2\) can be written as the disjoint union
\[
QT \cap Z_{\text{vis}}^2 = \bigcup_{k=1}^{\infty} (QT_k \cap Z_{\text{vis}}^2)
\]
we have that
\[
(25) = \sum_{k=1}^{\infty} \sum_{(a,b) \in QT_k \cap Z_{\text{vis}}^2} \kappa_1 \left( \frac{a}{Q}, \frac{b}{Q} \right) \kappa_2 \left( \frac{a}{Q}, \frac{b}{Q} \right) \kappa_3 \left( \frac{a}{Q}, \frac{b}{Q} \right)
\]
\[(26) = \sum_{k=1}^{\infty} k \sum_{(a,b) \in QT_k \cap Z_{\text{vis}}^2} \kappa_2 \left( \frac{a}{Q}, \frac{b}{Q} \right) \kappa_3 \left( \frac{a}{Q}, \frac{b}{Q} \right)
\]
\[(27) = \sum_{k=1}^{\infty} k \sum_{(a,b) \in QT_k \cap Z_{\text{vis}}^2} \kappa_2 \left( \frac{a}{Q}, \frac{b}{Q} \right) \kappa_3 \left( \frac{a}{Q}, \frac{b}{Q} \right)
\]

References


