A BRIEF SUMMARY OF PAST RESEARCH AND CURRENT PROJECTS

ALAN K. HAYNES

Much of our research is inspired by classical problems in number theory which are equivalent to the Riemann Hypothesis. One such equivalence arises from the elementary question, "How evenly spaced are the reduced fractions in the unit interval?" To make things precise define the Farey fractions of order Q by

$$\mathcal{F}_Q = \left\{ \frac{a}{q} \in (0,1] : 1 \leq q \leq Q, \gcd(a,q) = 1 \right\},$$

and enumerate them as $\mathcal{F}_Q = \{\gamma_1 < \gamma_2 < \ldots < \gamma_{N(Q)}\}$. Then it is known that the infimum of $\epsilon > 0$ for which

$$\sum_{n=1}^{N} \left(\gamma_n - \frac{n}{N}\right)^2 << Q^{2(\epsilon-1)} \tag{1}$$

as $Q \to \infty$ is equal to the supremum of real parts of the zeros of the Riemann zeta function (Fra24; Lan24).

Past Research

In 1971 M. Huxley proved that results analogous to (1) hold for the zero-free region of any Dirichlet L-function (Hux71). His results essentially state that the problem of determining the zero-free region of an L-function formed with a character of modulus d is equivalent to that of determining an upper bound for a weighted ℓ^2 discrepancy of the set

$$\mathcal{F}_{Q,d} = \left\{ \frac{a}{q} \in \mathcal{F}_Q : \gcd(q,d) = 1 \right\}.$$

Such subsets of the Farey fractions have been the object of a considerable amount of recent research (Hay03; BCZ03; CIZ03; Hay04; Hal05), both because of their relevance to GRH and because of known applications to particle physics (BGZ03a; BGZ03b). In (Hay03) we used well known estimates for Kloostermann sums (Wei48; Hoo57; Est61) to prove the following theorem in response to a question posed by A. Zaharescu.

Theorem (H.). *Fix a non-empty interval $[\alpha,\beta] \subseteq [0,1]$ and for $k, Q \in \mathbb{N}$ define*

$$\mathcal{F}_{Q,2,[\alpha,\beta]} = \mathcal{F}_{Q,2} \cap [\alpha,\beta],$$

$$N_{Q,2,[\alpha,\beta]}(k) = \#\left\{ \frac{a'}{q'} < \frac{a}{q} \text{ consecutive in } \mathcal{F}_{Q,2,[\alpha,\beta]} : q'a - a'q = k \right\},$$

and

$$\rho_{Q,2,[\alpha,\beta]}(k) = \frac{N_{Q,2,[\alpha,\beta]}(k)}{\#\mathcal{F}_{Q,2,[\alpha,\beta]}}.$$

Then for any $\epsilon > 0$ we have

$$N_{Q,2,[\alpha,\beta]}(k) = \left(\frac{2(\beta-\alpha)Q^2}{\pi^2}\right)\left(\frac{4}{k(k+1)(k+2)}\right) + O_\epsilon\left(Q^{3/2+\epsilon}\right)$$

and

$$\rho_{Q,2,[\alpha,\beta]}(k) = \frac{4}{k(k+1)(k+2)} + O_\epsilon\left(\frac{1}{Q^{1/2-\epsilon}}\right),$$

as $Q \to \infty$.

This is an interesting counterpart to the classical result that the numerator of the distance between consecutive elements of $\mathcal{F}_Q$ is always one (HW79). The joint distribution of $h-$tuples of numerators of differences in $\mathcal{F}_{Q,2}$ was later determined in (BCZ03). In (Hay04) we have extended the $h-$tuple result to $\mathcal{F}_{Q,p}$ where p is any prime number.

If d is not a prime power then a major obstruction in studying the set $\mathcal{F}_{Q,d}$ is the fact that two or more consecutive elements in $\mathcal{F}_Q$ may have denominators which are not coprime to d. We have recently found a way to attack this problem by considering a generalization of the Farey index defined by R. R. Hall and P. Shiu (HS03). The definitions and basic properties of our generalized Farey index can be found in (Hay06).

Current Projects

Our work with Farey fractions led us to explore an interesting collection of functions $f_\beta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, defined in (HV06a) and indexed by points β in $\mathbb{Q}/\mathbb{Z}$. As demonstrated in (Hat95), these functions form a complete orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$. In (HV06a) we have shown that they also encode information about continued fractions. Furthermore we have classified all orderings of the elements of $\mathbb{Q}/\mathbb{Z}$ for which these functions form a system of martingale differences. More precisely, we have proved the following result.

Theorem (H.-Vaaler). *If $\{\beta_1, \beta_2, \ldots\}$ is an ordering of the elements of $\mathbb{Q}/\mathbb{Z}$ then the sequence of functions $\{f_{\beta_n}\}_{n\in\mathbb{N}}$ forms a system of martingale differences (with respect to the canonical filtration of $\sigma-$algebras) if and only if the ordering $\{\beta_1, \beta_2, \ldots\}$ preserves the ancestry of the Stern-Brocot tree.*

Obviously the presence of martingales greatly simplifies questions about the almost everywhere convergence properties of orthonormal expansions in terms of the collection $\{f_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ (Gar70). By using a convergence theorem for a special collection of martingales studied by R. F. Gundy (Gun66) we have proved the following theorem.

Theorem (H.-Vaaler). *Let $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \cup \{\pm\infty\}$ be a Borel measurable function which is finite almost everywhere. Then there exists a countable collection of real numbers $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ such that for almost all irrational points α in $\mathbb{R}/\mathbb{Z}$ we have*

$$F(\alpha) = \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} q_{n-1}(\alpha) \sum_{m=1}^{a_n} c\left(\frac{mp_{n-1}(\alpha) + p_{n-2}(\alpha)}{mq_{n-1}(\alpha) + q_{n-2}(\alpha)}\right).$$

In the statement of this theorem $\alpha = [a_0; a_1, a_2, \ldots]$ denotes the regular continued fraction expansion of the irrational number α and $p_n(\alpha)/q_n(\alpha) = [a_0; a_1, \ldots, a_n]$ denotes the nth principal convergent to α. Since α is an element of $\mathbb{R}/\mathbb{Z}$ we ensure that these quantities are unique by setting $a_0 = 0$. The striking feature of this result is the weakness of the hypotheses on the function F, which is not even required to be integrable in order to draw the desired conclusion.

We are also interested in problems related to the Duffin-Schaeffer conjecture (Har98). The functions $\{f_\beta : \beta \in \mathbb{Q}/\mathbb{Z}\}$ together with maximal inequalities for partial sums of martingale differences can be used to give non-trivial upper bounds for the variance of a class of weighted sums of Duffin-Schaeffer type. In relation to this we have proved the following result, in which we write $h(\beta)$ for the order of β in the group $\mathbb{Q}/\mathbb{Z}$.

Theorem (H.-Vaaler). *Let $\{c(\beta) : \beta \in \mathbb{Q}/\mathbb{Z}\}$ be a collection of complex numbers and for each positive integer Q let $S_Q : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be defined by*

$$S_Q(x) = \sum_{\substack{\beta \in \mathbb{Q}/\mathbb{Z} \\ h(\beta) \leq Q}} c(\beta) f_\beta(x).$$

Then for each real number $1 < p < \infty$, and for each pair of positive integers $N \leq Q$ we have

$$\sum_{\substack{\beta \in \mathbb{Q}/\mathbb{Z} \\ h(\beta) \leq Q}} h(\beta)^{-1} \left(\max_{1 \leq R \leq N} |S_R(\beta)|^p \right) \leq \left(\frac{p}{p-1} \right)^p Q \int_0^1 |S_N(x)|^p \, dx.$$

In (HV06b) we demonstrate how this theorem may be applied to prove a wide variety of large sieve type inequalities.

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