Physics 162a, Fall 2009 – Problem Set 2

Due Wednesday, Sept. 30 in class

1. These exercises can be done easily using bra-ket notation.

   (a) The *trace* of an operator is the sum of its diagonal matrix elements, eg $\mathrm{Tr}\mathcal{O} = \sum_k \langle k|\mathcal{O}|k\rangle$. For operators $\mathcal{O}_{1,2}$, show that $\mathrm{Tr}\mathcal{O}_1\mathcal{O}_2 = \mathrm{Tr}\mathcal{O}_2\mathcal{O}_1$.

   (b) Show that $(\mathcal{O}_1\mathcal{O}_2)^\dagger = \mathcal{O}_2^\dagger\mathcal{O}_1^\dagger$.

   (c) Let $\mathcal{O}$ be a Hermitian operator with known eigenvalues and eigenvactors. Write $U = e^{i\lambda\mathcal{O}}$ in the form $\sum_k |k\rangle U_{kl}\langle l|$ for an appropriate basis $|k\rangle$ (which you should specify).

   (d) Consider the space of functions on the interval $[0, 1]$ with appropriate boundary conditions for the space to be a Hilbert space. (Which boundary conditions will not matter). For any orthonormal basis $|n\rangle$ on this space, compute $\sum_n \psi_n^*(x)\psi_n(y)$ where $\psi_n(x) = \langle x|n\rangle$.

2. Continuation of problem 6 from last week.

   (a) For a 3-point lattice, take the basis of eigenvectors of X. In this basis, write a Hermitian matrix P whose eigenvectors are $\tilde{f}_n$ with eigenvalues n. Compute $[X, P]$.

   (b) Prove that for a finite-dimensional vector space, there are no operators. A, B such that $[A, B] = \mathbf{1}$.

3. The Sturm-Liouville equation is:

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}f(x)\right) - q(x)f(x) - \lambda\rho(x)f(x)\ . \tag{1}$$

   Here $x, p(x), q(x), \lambda$ and $\rho(x)$ are all real. This is a generalization of an eigenvalue equation, where λ is the eigenvalue. Consider the Hilbert space of functions on the interval $[0, 2\pi]$ with Dirichlet boundary conditions. The Sturm-Liouville *operator* is:

$$\mathcal{L} = p(x)\frac{d^2}{dx^2} + p'(x)\frac{d}{dx} - q(x)\ . \tag{2}$$

(a) Prove that $\mathcal{L}$ is Hermitian.

If ρ is nonconstant, the theorem regarding Hermitian operators having an orthonormal basis of eigenfunctions generalizes. (Of course, we can always decompose with respect to the traditional eigenvectors of $\mathcal{L}$.)

(b) Assume that solutions to the Sturm-Liouville equation are distinct for each λ for which a solution exists. Prove that these solutions are orthogonal under the norm

$$\langle u|v\rangle_\rho = \int dx \rho(x) u^*(x) v(x) \ . \tag{3}$$