D. Transformation groups

(i) Definition: a group (G, *) is a set of elements g, g', g''... together with a (not necessarily commutative) multiplication rule

\[ g_1 * g_2 \in G \]

such that:

* Multiplication is associative

\[ g_1 (g_2 * g_3) = (g_1 * g_2) * g_3 \]

* There exists a unit e such that \( g * e = e * g = g \) for all \( g \) in \( G \)

* For each \( g \) in \( G \), there exists an inverse \( g^{-1} \) such that

\[ g * g^{-1} = e = g^{-1} * g \]

Clearly, the set of transformations forms a group.

Example: cyclic group of \( n \) elements

\[ g : \{1, 2, ..., n\} \rightarrow \{n, 1, 2, ..., n\} \]

Clearly commutative

Example: permutation group of \( n \) elements

Generated by \( \sigma \) and \( \tau \).

Note that \( \sigma \tau \sigma^{-1} \neq \tau \sigma \).

Example: rotation in \( \mathbb{R}^3 \) (clearly)

Example: Parity: \( P(x) = 1 - x \), \( P^2 = -P \)
(2) **Representations.**

(a) **Definition:** A unitary representation of $G$ on a Hilbert space $H$ is a map

$$\phi: G \rightarrow \text{unitary operators on } H$$

such that

- $\phi(g_1 g_2) = \phi(g_1) \phi(g_2)$
- $\phi(e) = I$

(b) A representation on $H$ is reducible if $H$ can be written as a direct sum of subspaces $H_1, H_2, \ldots, H_k$

$$H = \bigoplus_{i=1}^{k} H_i$$

such that $\phi(g): H_i \rightarrow H_i \quad \forall g$

"invariant subspace"

(c) A representation on $H$ is irreducible if it has no invariant subspace.

**Example:** $D_3$ - Hilbert space of states with angular momentum $\frac{1}{2}$ basis $|ij\rangle \sim a_{ij}\chi_{ij}$

- Finite rotation about $x,y,z$ axes: $e$
- Inversion: $i = \frac{1}{\sqrt{3}}[i\delta_{ij} + \delta_{ij}]$
- $J_z = J_x, J_y, J_3$: $(|j,m\rangle \rightarrow i^m |j,m\rangle)$

- Classic example of decomposition into irreducible representations.

$$D_3 \otimes D_3 = D_6 \otimes D_6$$

Reducible, Irreducible
Continuous groups and infinitesimal transformations

- A "continuous group" $G$ is one for which the group elements can be labelled by a set of continuous parameters:

$$g = g(t_1, ..., t_n)$$

- Such a group could have multiple components, e.g., $O(3)$'s rotations $R(3)$ & inversion $j$: $\mathbb{R} \to \mathbb{R}$

Usually, in component connected to the identity, we take $g(t_1, ..., t_n) = 1$

- Infinitesimal transformations

Let us focus on a single parameter $t$:

$$g(t) = 1$$

$$\epsilon \ll g(t) \approx g(0) + t \delta g + \cdots$$

$$= 1 - i t \delta g (\delta g)^* + ...$$

$$\to \mathbb{C}^d: \quad g^+(1) g(0) = \mathbb{1} + i t (\delta g)^* - i t \delta g = \mathbb{1}$$

$$\Rightarrow$$ infinitesimal (unitary) transformations generated by Hermitean operators

$$\delta g \delta^* = (1 - \delta g^2) \delta + i \delta^* \delta = 0 - i \delta \delta^*$$

$$= 0$$

where $t$ is any parameter
Examples

(i) Translation: \( \hat{g}(a) = e^{-i \lambda a} \)  \( \hat{g} = \hat{\rho} \)

(ii) Rotations: \( \hat{g}_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \)

(iii) Time translations:

\[ U \cdot T = e^{-i \sigma \theta (\mathbf{H} \cdot t)} \]

\[ H(\mathbf{t}) \]
III. Symmetries

A. Definition

- A symmetry (let the Hamiltonian) is a transformation \( \mathbf{\hat{A}} \) such that
\[
\mathbf{\hat{A}}^\dagger \mathbf{H} \mathbf{\hat{A}} = \mathbf{H}
\]
(i.e., \( \mathbf{H} = \mathbf{H}(\mathbf{\hat{A}}) \), should be for \( \forall \mathbf{\hat{A}} \))

- Given a continuous family of symmetries \( \mathbf{\hat{\alpha}}(t) \), \( \mathbf{\hat{\alpha}}(0) = \mathbf{I} \), \( \mathbf{\hat{\alpha}}(t, \mathbf{H}) \)
\[
\Rightarrow [\mathbf{\hat{\alpha}}(t), \mathbf{H}] = 0
\]

- Obvious examples:
\[
\begin{align*}
\mathbf{H} &= \frac{1}{2m} \mathbf{\hat{p}}^2 + \frac{1}{2m} v(x)^2 \\
\mathbf{\hat{\alpha}}(t) &= \mathbf{\hat{R}}(\theta) \text{ rotations}
\end{align*}
\]
\[
\mathbf{V}(\mathbf{\hat{\alpha}}) = 0 \quad \text{if} \quad \mathbf{\hat{R}}(\theta) \text{ (translation)}
\]

B. Conservation laws

(i) Quantum mechanics

- Given \( [\mathbf{\hat{\alpha}}, \mathbf{H}] = 0 \), we can diagonalize \( \mathbf{\hat{\alpha}}, \mathbf{H} \) simultaneously.

- Choose \( \mathbf{\hat{\alpha}} \) such that \( \mathbf{\hat{\alpha}} \mathbf{\hat{\psi}} = \mathbf{\hat{\psi}} \)
\[
\mathbf{\hat{\alpha}} \mathbf{\hat{\psi}} = \mathbf{\hat{\psi}} \quad \Rightarrow \quad \mathbf{\hat{\alpha}} \mathbf{\hat{\psi}} = \mathbf{\hat{\psi}} \quad \Rightarrow \quad \mathbf{\hat{\psi}} \in \text{"conserved quantity"}
\]

Symmetries \( \Rightarrow \) conservation laws
Example: if $T$ is a symmetry, momentum is conserved.

If rotations are a symmetry, angular momentum is conserved.

If time translations are a symmetry, energy is conserved.

(1) Classical mechanics: Noether's Theorem

Let $L = L(q^i, \dot{q}^i)$.

$$S = \int dt \left( \frac{d}{dt} L(q^i, \dot{q}^i) - \frac{d}{dt} L(q^i, \dot{q}^i) \right)$$

$s = \int dt L \implies S = 0$ = Euler-Lagrange equation

Let $S$ be invariant under classical transformations $q^i \to q^i(t) + \delta q^i(t)$

This defines a classical symmetry.

$$\delta L = \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i$$

$$= \sum_i \left( \delta q^i \frac{\partial L}{\partial \dot{q}^i} + \delta \dot{q}^i \left( \frac{\partial L}{\partial q^i} \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right)$$

but $\delta \dot{q}^i = 0 \implies$ Euler-Lagrange equations are the equations of motion.

$\delta L = 0$ are the equations of motion.

Therefore, $\delta L = 0$ is a conserved quantity

$Q = \sum_i \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \wedge \dot{q}^i$ is a conserved quantity.