1. **Mass-radius relationship for white dwarfs.** (From Kittel and Kroemer, *Thermal Physics*). Consider a white dwarf of mass $M$ and radius $R$. Let the electrons be degenerate (e.g., in their ground state) but nonrelativistic; the protons are nondegenerate.

(a) Argue that the order of magnitude of the gravitational self-energy is $-G_N M^2 / R$. (If the mass density is constant, the exact potential energy is $-3G_N M^2 / (5R)$.)

(b) Show that the order of magnitude of the kinetic energy of the electrons in their ground state is:

$$\frac{\hbar^2 N^{5/3}}{mR^2} \approx \frac{\hbar^2 M^{5/3}}{mM_H^{5/3} R^2}$$  \hspace{1cm} (1)

where $m$ is the electron mass and $M_H$ the proton mass.

(c) Assuming (as required by the virial theorem) that the gravitational and kinetic energies are of the same order of magnitude, show that $M^{1/3} R \approx 10^{20} (\text{gm})^{1/3} - \text{cm}$.

(d) If the mass is 1 solar mass ($2 \times 10^{33}\text{gm}$), what is the density of the white dwarf?

(e) It is believed that pulsars are stars made up of a cold degenerate gas off neutrons. Show that for a neutron star, $M^{1/3} R \approx 10^{17} (\text{gm})^{1/3} - \text{cm}$. What is the radius for a 1-solar-mass neutron star, in kilometers?

2. A homogenous system of spin-0 particles in a cubic box of size $L$, as $L \to \infty$, is described in the second-quantized description by the Hamiltonian

$$H = \frac{\hbar^2}{2m} \int d^3x \vec{\nabla} \hat{\phi}^+(x) \cdot \vec{\nabla} \hat{\phi}^-$$

$$+ \frac{1}{2} \int \int d^3xd^3x' \hat{\phi}^+(x) \hat{\phi}^+(x') V(\vec{x} - \vec{x}') \hat{\phi}^-(x') \hat{\phi}^-(x)$$  \hspace{1cm} (2)

We will treat $V$ as a perturbation.
(a) Show that the expectation value of the interacting Hamiltonian, in the ground state of N bosons for the noninteracting system, is

\[ E^{(1)}_N = \frac{(N - 1)\tilde{V}(0)}{2L^3} \]  

where

\[ V(\vec{x}) = \int d^3k \frac{1}{(2\pi)^3} \tilde{V}(k)e^{i\vec{k} \cdot \vec{x}} \]

The factor of \( L^3 \) is a bit tricksy. It will show up via the normalization of the states. If you express the ground state in terms of some momentum basis, the norm of this states will have a bunch of delta functions in momentum space with argument 0. These are \( \delta^{(3)}(k - k) = \delta^{(3)}(k) \) and you can set that delta function equal to \( L^3 \) by dimensional analysis. Next, since the state itself has norm proportional to \( L^{3N} \) rather than to 1, the expression for the energy shift to first order in perturbation theory is:

\[ \delta E^{(1)} = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \]

which is dimensionally correct. Alternatively, you can re-develop the formalism working at finite \( L \) always. Either of these are acceptable, but either way you should explain your reasoning.

(b) Show that the second-order contribution (in \( V \)) to the ground-state energy is:

\[ E^{(2)}_N = -\frac{N - 1}{L^3} \int d^3q \frac{\vert \tilde{V}(q) \vert^2}{(\hbar^2 q^2)/m} \approx -\frac{n}{2} \int d^3q \frac{\vert \tilde{V}(q) \vert^2}{(\hbar^2 q^2)/m} \]

where \( n = N/L^3 \) is the particle density.

(c) If \( V(|x|) < 0 \) for all \( \vec{x} \), and if \( \int d^3x |V(x)|^2 < \infty \), show that the system will collapse (Hint: start from \( E^{(1)}_N / N \) as a function of the density).