

Physics 162b – Quantum Mechanics, winter 2009

Problem Set 1

Due at the beginning of class on Friday, Feb 27.

For the final 3 problems you should read chapter 11 of Baym, even if we do not get all the way through the material in class.

1. Let us now examine a particle with charge e in *two* dimensions, with coordinates (x^1, x^2) , in a constant magnetic field. The magnetic field is defined from a vector potential as:

$$B = \partial_1 A_2 - \partial_2 A_1 \quad (1)$$

where $\partial_i = (\partial/\partial x^i)$. For constant B , we can choose A (this involves a particular choice of gauge) as follows:

$$\begin{aligned} A_1 &= -\frac{1}{2} B x^2 \\ A_2 &= \frac{1}{2} B x^1 \end{aligned} \quad (2)$$

- (a) Write the Hamiltonian for the particle. (This is supposed to be an easy step, just to make sure you understood the lectures).
- (b) Write the coordinates in dimensionless form, $x^i = \ell y^i$. Rewrite the time-independent Schrödinger equation in terms of these dimensionless coordinates. Choose ℓ so that all dimensionful quantities appear in terms of an energy ϵ_0 which appears in front of H :

$$H = \epsilon_0 h . \quad (3)$$

Find ϵ_0 . The time-independent Schrödinger equation, or the eigenvalue equation for the Hamiltonian, can then be written as:

$$h|\psi\rangle = \frac{E}{\epsilon_0}|\psi\rangle = \epsilon|\psi\rangle \quad (4)$$

In a coordinate basis, the wavefunction will have the form $\psi(y) = \psi(x/\ell)$. ℓ is the *magnetic length* and is the characteristic length scale for the problem. ϵ_0 is the characteristic energy scale. You may choose to work in dimensionless coordinates, or not, for the remainder of this problem.

- (c) re-write the Hamiltonian in complex coordinates: $z = \frac{1}{2}(x^1 + ix^2)$, $\bar{z} = \frac{1}{2}(x^1 - ix^2)$.
- (d) Using the formula for the transformation of the vector potential A_i from variables x^i to variables t^a is:

$$A_a = \sum_i \frac{\partial x^i}{\partial t^a} A_i \quad (5)$$

compute $A_z, A_{\bar{z}}$.

- (e) The *covariant derivative* acting on the wavefunction for a charged particle is:

$$D_i = \partial_i - i \frac{e}{\hbar c} A_i . \quad (6)$$

In class I called it Π . Compute $[D_i, D_j]$ for *general* A_i , and for the gauge field (2). Also, compute $[D_z, D_{\bar{z}}]$ and $D_z, D_{\bar{z}}$. Write the Hamiltonian in terms of D_i . (Yes, its that stupid. Just making sure you know where you are.) Rewrite it in terms of D_z and $D_{\bar{z}}$.

- (f) Based on the form of the Hamiltonian in complex coordinates, and the commutation relations for D_z and $D_{\bar{z}}$, what are the energy eigenvalues? Hint: think about the simple harmonic oscillator. This should all look horribly familiar. Note that the Hamiltonian will have a degenerate spectrum.
- (g) For particles in \mathcal{R}^n , the rotation operator is:

$$L_{ij} = \hat{x}^i \hat{p}_j - \hat{x}^j \hat{p}_i \quad (7)$$

Obviously for \mathcal{R}^2 , only $\hat{L} = L_{12} = -L_{21}$ counts. Show that this is the infinitesimal generator of rotations about the origin by studying its actions on $\psi(x^1, x^2)$.

- (h) Write \hat{L} in complex coordinates. Now, a wavefunction near the origin should be single-valued. That is, as a function of z, \bar{z} , you should be able to define it on the plane without branch cuts. Write a general expression for the single-valued eigenfunctions of \hat{L} as a function of z, \bar{z} . What are the eigenvalues? **Hint:** you may express $\psi(z, \bar{z})$ in a power series near the origin.
- (i) Write the Hamiltonian in the basis x^1, x^2 so that it is clearly the sum of two harmonic oscillator Hamiltonians and the rotation operator. Show that the rotation operator commutes with the sum of

the two Harmonic oscillator Hamiltonians. Using this fact, write the energy spectrum in terms of eigenvalues n_1, n_2 of the harmonic oscillator Hamiltonians and the eigenvalues l of \hat{L} (which we will compute explicitly below). Express it in terms of the characteristic scales.

- (j) The *Lowest Landau Level* (LLL) of the system is the set of all states for which $l = n_1 + n_2$. Write the energy of the states in the lowest Landau level. Using this, and the form of the Hamiltonian in complex coordinates, show that the LLL states satisfy $D_{\bar{z}} = 0$. Show that the general form of the wavefunction is the product of an analytic function (an analytic function is a function of z and *not* of \bar{z}). Write down a basis for the LLL in terms of eigenfunctions of \hat{L} . Now we have split the degeneracy of the lowest Landau level.
- (k) Compute $\langle (x^1)^2 + (x^2)^2 \rangle$ for all eigenkets of \hat{L} in the lowest Landau level. Express it in terms of the characteristic scales.

2. Tensor operators and the Wigner-Eckart theorem.

- (a) As much as possible, relate the matrix elements

$$|n', l', m'\rangle \langle \mp \frac{1}{\sqrt{2}} \rangle (x \pm iy) \langle n, l, m| \quad (8)$$

and

$$|n', l', m'\rangle \langle z \rangle \langle n, l, m| \quad (9)$$

using only the Wigner-Eckhart theorem. State under which conditions (eg on the quantum numbers) the matrix elements can be non-vanishing.

- (b) Write xy , xz , and $(x^2 - y^2)$ as components of an irreducible spherical tensor of rank 2.
- (c) For a charged particle, $Q \equiv e|\alpha, j, m = j\rangle (3z^2 - r^2) \langle \alpha, j, m = j|$, with e the electric charge is the *quadrupole moment*. Evaluate

$$e|\alpha, j, m'\rangle \langle x^2 - y^2 \rangle \langle \alpha, j, m = j| \quad (10)$$

in terms of Q and appropriate Clebsch-Gordan coefficients.

3. Baym, Chapter 7, problem 4

4. Consider a symmetric rectangular double-well potential:

$$V = \begin{cases} \infty & |x| > a + b \\ 0 & a < |x| < a + b \\ V_0 > 0 & |x| < a \end{cases} \quad (11)$$

Assume V_0 is large compared to the quantized energies of the low-lying states. Obtain an approximate expression for the energy splitting between the two lowest-lying states.

5. Consider the following Hamiltonian for a spin-one system:

$$H = A\hat{S}_z^2 + B(\hat{S}_x^2 - \hat{S}_y^2) \quad (12)$$

Solve this problem exactly to find the normalized energy eigenstates and eigenvalues. Is this invariant under time reversal? How do the normalized energy eigenstates behave under time reversal?

6. (17.2.1 of Shankar, with a slight modification of notation). Consider the Hamiltonian

$$H_0 + \epsilon H_1 \quad (13)$$

where

$$H_0 = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \quad (14)$$

is the SHO Hamiltonian and

$$H_1 = \lambda\hat{x}^4 \quad (15)$$

- (a) Find the leading correction (first order in perturbation theory) to the energy for energy level n .
- (b) Give a good argument that no matter how small ϵ is, the perturbation expansion will break down for large enough n . Find the value of n for fixed ϵ .

7. Baym, problem 11.2

8. Baym, problem 11.4

9. Baym, problem 11.5