The \( q \)-Catalan numbers 1,1+\( q \),1+2+\( q^2 \)+\( q^3 \),1+3+3\( q^2 \)+3\( q^3 \)+2\( q^4 \)+\( q^5 \)+\( q^6 \),\ldots, which have been introduced by Carlitz and Riordan ([3], cf. also[8] and [11]), are defined by

\[
C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-k-1}(q)
\]

with initial value \( C_0(q) = 1 \). They are polynomials in \( q \) of degree \( \left\lfloor \frac{n}{2} \right\rfloor \).

Let \( f(z,q) = \sum_{k=0}^{\infty} C_k(q) z^k \) be their generating function, which is uniquely determined by the functional equation \( f(z,q) = 1+zf(z,q)f(qz,q) \). This implies the well known fact that it can be represented in the form

\[
f(z,q) = \frac{E_z(-q z)}{E_z(-z)}, \tag{1}
\]

where \( E_r(z) \) denotes the generalized \( q \)-exponential function

\[
E_r(z) = \sum_{k=0}^{\infty} \binom{n}{2} \left( \frac{z}{1-q}(1-q^2)\cdots(1-q^k) \right).
\]

As shown in [8] the related polynomials \( q^{\binom{n}{2}} C_n \left( \frac{1}{q} \right) \), which are given by

\[
1,1+q,1+q+2q^2+q^3,1+q+2q^2+3q^3+3q^4+q^5,\ldots
\]

have a nice combinatorial interpretation which implies that \( q^{\binom{n}{2}} C_n \left( \frac{1}{q} \right) = 1+p(1)q+\cdots+p(n-1)q^{n-1}+O(q^n) \), where \( p(n) \) denotes the number of partitions of the number \( n \).

This may also be seen from (1): If we compare the coefficients of \( z^n \) in

\[
f(z,q)E_z(-z) = E_z(-qz), \text{ replace } q \text{ by } \frac{1}{q} \text{ and multiply both sides with } q^{\binom{n}{2}} \text{ we get}
\]

\[
\sum_{k=0}^{n} \binom{n}{2} \frac{q^{k(k-3)}}{(1-q)^{\frac{k(k-3)}{2}}} C_{n-k} \left( \frac{1}{q} \right) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}.
\]

But the left-hand side is of the form

\[
q^{\binom{n}{2}} C_n \left( \frac{1}{q} \right) + O(q^n) \quad \text{and the right-hand side may be written as}
\]

\[
\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} + O(q^n) = \sum_{k=0}^{\infty} p(k)q^k + O(q^n).
\]
In this note we want to sketch some extensions of (1).

a) First observe that

\[ E_r(z) - E_r(qz) = zE_r(q^r z). \]  

(3)

For \( n \in \mathbb{N} \) define

\[ G_r(z,n) = \sum_{k \geq 0} G(k,n,r)z^k := \frac{E_r(-q^n z)}{E_r(-z)}. \]  

(4)

Then we have

\[ G_r(z,n+1) = G_r(z,n) + q^n zG_r(z,n+r). \]  

(5)

Comparing coefficients we get

\[ \frac{G(k,n+1,r) - G(k,n,r)}{q^n} = G(k-1,n+r,r) \]  

(6)

with \( G(k,0,r) = [k = 0] \) and \( G(0,n,r) = 1 \).

This implies

\[ G_r(z,1) = 1 + zG_r(z,r). \]  

(7)

These are the characteristic properties of the \( q \)–Gould polynomials (cf. [5]).

For \( q = 1 \) they have the explicit formula \( G(k,n,r) = \frac{n}{n+r} \binom{n+rk}{k} \) (cf. e.g. [10]).

For general \( q \) special values are \( G(k,n,0) = q^k \binom{n}{k} \) and \( G(k,n,1) = \binom{n+k-1}{k} \), where \( \binom{n}{k} \) denotes a \( q \)–binomial coefficient. For \( r > 1 \) no explicit formulas are known.

Note that \( G_2(z,1) = 1 + zG_2(z,2) = 1 + zG_2(z,1)G_2(qz,1) = f(z,q) \) is the generating function of the \( q \)–Catalan numbers.

From

\[ \frac{E_r(-q^n z)}{E_r(-z)} = \frac{E_r(-qz)}{E_r(-z)} \frac{E_r(-q^2 z)}{E_r(-rz)} \cdots \frac{E_r(-q^n z)}{E_r(-q^{n-1} z)} \]

we get

\[ G_r(z,n) = G_r(z,1)G_r(qz,1) \cdots G_r(q^{n-1} z,1) \]  

(8)

and

\[ G_r(z,m+n) = G_r(z,m)G_r(q^m z,n). \]  

(9)
b) Let now \((a + b)^k := (a + b)(a + q b) \cdots (a + q^{k-1} b)\)
and consider the modified exponential function

\[
h(z, a, b, q) = \sum_{k \geq 0} q^{k \choose 2} \frac{(a + b)^k}{(1 - q)^k} (-z)^k,
\]

which for \((a, b) = (0, 1)\) reduces to \(E_z(-z)\).

We want to study the series

\[
f(z, a, b, q) = \frac{h(qz, a, b, q)}{h(z, a, b, q)}.
\]

From

\[
h(z, a, b, q) - h(qz, a, b, q) = \sum_{k \geq 0} q^{k \choose 2} \frac{(a + b)^k}{(1 - q)^k} (-z)^k (1 - q^k) =
\]

\[
= -z \sum_{k \geq 0} q^{k \choose 2} \frac{(a + b)^k}{(1 - q)^k} (a + q^{k-1} b) (-qz)^{k-1}
\]

we deduce

\[
h(z, a, b, q) - h(qz, a, b, q) = \sum_{k \geq 0} q^{k \choose 2} \frac{(a + b)^k}{(1 - q)^k} (-z)^k (1 - q^k) =
\]

\[
= -z \sum_{k \geq 0} q^{k-1 \choose 2} \frac{(a + b)^{k-1}}{(1 - q)^{k-1}} (a + q^{k-1} b) (-q)^{k-1} = -azh(qz, a, b, q) - bzh(q^2 z, a, b, q)
\]
i.e.

\[
f(z, a, b, q) = 1 + azf(z, a, b, q) + bzf(z, a, b, q) f(qz, a, b, q)
\]

and

\[
h(z, a, b, q) - h(qz, a, b, q) = -z \sum_{k \geq 0} q^{k-1 \choose 2} \frac{(a + b)^{k-1}}{(1 - q)^{k-1}} (a + q^{k-1} b) (-qz)^{k-1} = -(a + b)zh(qz, a, qb, q).
\]

Let now

\[
g(z, a, b, q) = \frac{h(qz, a, qb, q)}{h(z, a, b, q)}.
\]

Then

\[
f(z, a, b, q) = 1 + (a + b)zg(z, a, b, q).
\]

This implies

\[
\frac{g(qz, a, b, q)}{g(z, a, b, q)} = \frac{h(q^2 z, a, qb, q)}{h(qz, a, b, q)} \frac{h(z, a, b, q)}{h(qz, a, qb, q)} = \frac{f(qz, a, qb, q)}{f(z, a, b, q)}.
\]
If we write
\[ f(z,a,b,q) = \sum_{n \geq 0} C_n(a,b,q)z^n, \]
then \( C_n(a,b,q) \) is a \( q \)-analogue of \( C_n(a,b) = \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k-1} b^{n-k} (a+b)^k \).

For it is easy to verify that the uniquely determined formal power series \( f(z,a,b) \), which satisfies the functional equation
\[ f(z,a,b) = 1 + azf(z,a,b) + bzf(z,a,b)^2 \]
has the series expansion
\[ f(z,a,b) = 1 + \sum_{n \geq 1} \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} b^{n-k} (a+b)^k z^n, \]  

The numbers \( N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1} = \binom{n}{k} \binom{n-1}{k-1} - \binom{n}{k-1} \binom{n}{k} \) are called Narayana numbers. (cf. e.g. [11] or [2]). The first terms of this sequence are

\[
\begin{array}{cccccccccc}
1 & & & & & & & & & \\
1 & 1 & & & & & & & & \\
1 & 3 & 1 & & & & & & & \\
1 & 6 & 6 & 1 & & & & & & \\
1 & 10 & 20 & 10 & 1 & & & & & \\
1 & 15 & 50 & 50 & 15 & 1 & & & & \\
\end{array}
\]

By choosing \( a = s - 1 \) and \( b = 1 \) and setting \( F = f(z,s-1,1) - 1 \) we get the well-known fact (cf. [11]) that the generating function \( F(z,s) = \sum \sum N(n,k)s^k z^n \) satisfies
\[ zF(z,s) + (sz + z - 1)F(z,s) + sz = 0. \]  

It is easily verified that \( G(z,s) = \frac{Sz}{1-z} + \frac{s}{1-z} F(\frac{Sz}{1-z},z) \) satisfies the same equation and therefore \( G(z,s) = F(z,s) \). By comparing coefficients this implies that
\[ \sum_{n \geq 1} N(n,k)z^n = \sum_{j=1}^{k-1} N(k-1,j)z^{k-1+j} \frac{z^{k-1} + j}{(1-z)^{2k-1}}. \]

This is a refinement of the trivial fact that \( \Delta^{2k-1} N(n,k) = (E - 1)^{2k-1} N(n,k) = 0 \) if we denote by \( E \) the shift operator defined by \( Ef(n) = f(n+1) \) and by \( \Delta = E - 1 \) the difference operator.

It is clear that \( N(n,k) = N(n,n+1-k) \). This can also be seen from the fact that \( sF\left(\frac{sz}{s},\frac{1}{s}\right) \) satisfies (18) too.

Of course \( C_n(0,1) = \frac{1}{n+1} \binom{2n}{n} = C_n \) are the well known Catalan numbers.
We call $C_n(a,b,q)$ a $q$–Narayana polynomial.

$C_n(a,b,q)$ satisfies

$$C_n(a,b,q) = (a+b)C_{n-1}(a,b,q) + b\sum_{k=1}^{n-1} q^k C_k(a,b,q) C_{n-k-1}(a,b,q)$$  \hspace{1cm} (19)$$

with initial value $C_0(a,b,q) = 1$.

The first values are $1, a+b, a^2 + (2+q)ab + (1+q)b^2, \ldots$.

From the definition is clear that $C_n(a,b,q)$ is a polynomial in $a,b$ which is homogeneous of degree $n$. Therefore $C_n(a,b,q)$ has for $n \geq 1$ a unique representation in the form

$$C_n(a,b,q) = \sum_{k=0}^{n} N(n,k,q)(a+b)^k b^{n-k}.$$  \hspace{1cm} (20)$$

In order to study the $q$–Narayana numbers $N(n,k,q)$ we may choose $a = s-1, b = 1$.

Then we get

$$C_n(s-1,1,q) = \sum_{k=1}^{n} N(n,k,q) s^k$$  \hspace{1cm} (21)$$

and the recurrence

$$C_n(s-1,1,q) = sC_{n-1}(s-1,1,q) + \sum_{k=1}^{n-1} q^k C_k(s-1,1,q) C_{n-k-1}(s-1,1,q).$$  \hspace{1cm} (22)$$

The $q$–Narayana numbers are polynomials in $q$ with integer coefficients.

The first values of these $q$–Narayana numbers are given in the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^2$</td>
<td>2 $q + q^2$</td>
</tr>
<tr>
<td>$q^3$</td>
<td>$q^2 + 2 q^3 + 2 q^4 + q^5$</td>
</tr>
<tr>
<td>$q^4$</td>
<td>2 $q^3 + 3 q^4 + 2 q^5 + 2 q^6 + q^7$</td>
</tr>
</tbody>
</table>

Some $q$–Narayana numbers can be explicitly given. E.g. $N(n,1,q) = \binom{n}{k}$ for $n \geq 1$.

Comparing the coefficients of $s^{n-2}$ we get

$$N(2n+1,2,q) = q^{2n} N(2n,2,q) + q \sum_{k=1}^{2n-1} q^k \binom{k}{2} \binom{2n-k}{2} = q^{2n} N(2n,2,q) + 2q^{n} \sum_{i=1}^{n} q^{i-n},$$

because $\binom{k+1}{2} + \binom{2n-k}{2} = n^2 + 2\binom{n-k}{2}$.

In the same way we get

$$N(2n,2,q) = q^{2n-1} N(2n-1,2,q) + q^{n-2} \left( 1 + 2 \sum_{i=1}^{n-1} q^i \right).$$

This implies that $N(n,2,q)$ satisfies the recurrence

$$(E^4 - q^{n+2} (1+q) E^3 + q^{n+2} (q^{n+2} - 1) E^2 + 2q^{2n+3} E - q^{3n+3}) N(n,2,q) = 0.$$
Let now $M(n,k,q) = N(n,n+1-k,1)$. Then $M(n,1,q) = 1$ and

$$M(n,2,q) = q^{n-1} + 2q^{n-2} + \cdots + (n-1)q.$$ The last one follows from (22) by comparing coefficients of $s$, which gives

$$M(n,2,q) = M(n-1,2,q) + \sum_{k=1}^{n-1} q^k.$$ The first one by comparing the coefficients of $s^{n-1}$.

We get $(E-1)M(n,2,q) = q[n]$ and therefore $(E-1)(E-q)M(n,2,q) = q$. This implies the homogeneous recurrence $(E-1)^2(E-q)M(n,2,q) = 0$.

Computer experiments suggest the following facts:

1) For each $k$

$$(E-1)^k(E-q)^{k-1}(E-q^2)^{k-2} \cdots (E-q^{k-1})M(n,k,q) = 0$$

and more generally

$$(E-1)^{k-1}(E-q)^{k-1}(E-q^2)^{k-2} \cdots (E-q^{k-1})M(n,k,q) = q^{k-1}(1-q^2)^{k-2}(1-q^3)^{k-3} \cdots (1-q^{k-1}).$$

2) $\deg(N(n,k,q)) = \frac{(n-k)(n+k-1)}{2}$ for $n \geq k$ and

$$\frac{(n-k)(n+k-1)}{2} N(n,k,q) = \frac{p(k,2)}{q} + jk - 1 q^j + O(q^{n-k+1}),$$

where $p(j,k-1)$ denotes the number of partitions of the number $j$ with precisely $k-1$ different parts. Of course $p(j,2) = d(j)$ the number of divisors of $j$. (Cf. The On-Line Encyclopedia of Integer Sequences OEIS A060177).

3) $\deg(N(n,n-k,q)) = \frac{k(2n-k-1)}{2} = nk - \frac{k+1}{2}$ for $n \geq k+1$ and

$$\frac{k(2n-k-1)}{2} N(n,n-k,q) = \frac{p(k,2)}{q} + jk - T(k^2 + j,k) q^j + O(q^{n-k+1}),$$

where $T(n,k)$ denotes the number of partitions of $n$ with Durfee square of size $k$. (Cf. OEIS A115994).

Till now I have no proof of these results.

It is well known (cf. e.g. [11]) that the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ can be characterized by the values of their Hankel determinants. They satisfy $\det\binom{C_{i+j}}{i,j=0}^n = 1$ and $\det\binom{C_{i+j+1}}{i,j=0}^n = 1$.

We want to show that the $q$–Narayana polynomials can also be characterized by the values of their Hankel determinants. This shows that these polynomials are in some sense a natural generalization of the $q$–Catalan numbers.
Theorem
The polynomials $C_n(a,b,q)$ are characterized by their Hankel determinants
\[
\det(C_{i,j}(a,b,q))_{i,j=0}^n = q^{\binom{n+1}{2}} b^{\binom{n+1}{2}} (a+b)^n (a+q^2b)^{n-1} \cdots (a+q^{n-1}b) 
\] (23)
and
\[
\det(C_{i,j+1}(a,b,q))_{i,j=0}^n = q^{\binom{n+1}{2}} b^{\binom{n+1}{2}} (a+b)^{n+1} (a+q^2b)^n \cdots (a+q^n b). 
\] (24)

Remark
For the $q$–Catalan numbers this reduces to
\[
\det(C_{i,j}(q))_{i,j=0}^n = q^{\binom{n+1}{2}} b^{\binom{n+1}{2}} 
\] and
\[
\det(C_{i,j+1}(q))_{i,j=0}^n = q^{\binom{n+1}{2}} b^{\binom{n+1}{2}} .
\]
These special cases have been proved by another method in [6].

In order to prove the theorem we need the following well-known (cf. e.g. [1] or [9])

Lemma
Let
\[
\sum_{k=0}^{\infty} \mu_k z^k = \frac{1}{1-s_0 z - t_0 z^2 - \cdots - t_1 z^2} \]
\[
= \frac{1}{1-s_0 z - t_1 z^2 - \cdots} 
\] (25)

Then the Hankel determinants have the following values
\[
\det(\mu_{i,j})_{i,j=0}^n = t_0 t_1 \cdots t_{n-1} 
\]
and
\[
\det(\mu_{i,j+1})_{i,j=0}^n = d_n \cdots t_{n-1}, 
\]
where $d_n = s_{n-1} d_{n-1} - t_{n-2} d_{n-2}, d_0 = 1, d_1 = s_0$.

Let $F$ be the linear functional defined by $F(z^n) = \mu_n$.

Then the polynomials $p_n(z)$, defined by
\[
p_0(z) = 1, p_1(z) = z - s_0, 
\]
and $p_k(z) = (z - s_{k-1}) p_{k-1}(z) - t_{k-2} p_{k-2}(z)$
satisfy $F(p_n p_m) = t_0 \cdots t_{n-1} [n = m]$, 
\text{i.e. are orthogonal with respect to the linear functional $F$.}

Proof of the Theorem
Using (14) we can write (12) as
\[
f(z,a,b,q) = 1 + (a+b)zf(z,a,b,q) + (a+b) bq z^2 f(z,a,b,q)g(qz,a,b,q), 
\] (26)
where the series $g(z,a,b,q)$ satisfies
\[
g(z,a,b,q) = 1 + (a+b) zg(z,a,b,q) + q bq(z,a,b,q) + qb(a+b) z^2 g(z,a,b,q). 
\]
This is equivalent with
\[ g(z,a,b,q) = 1 + (a+b+qb)\zg(z,a,b,q) + q^2b(a+qb)z^2g(z,a,b,q)g(qz,a,qb,q). \] (27)

For (15) implies
\[
q\zg(qz,a,b,q)(1+(a+b)\zg(z,a,b,q)) = q\zg(qz,a,b,q)f(z,a,b,q)
\]
\[
= q\zg(z,a,b,q)f(qz,a,qb,q) = q\zg(z,a,b,q)(1+(a+qb)\zg(qz,a,qb,q)).
\]

Now (27) is equivalent with
\[
g(z,a,b,q) = \frac{1}{1-s_1z-t_1z^2g(qz,a,qb,q)}
\]
with \(s_i = (a+b+qb)\) and \(t_1 = q^2b(a+qb)\).

This gives us a representation of \(g(z,a,b,q)\) as a continued fraction of the form
\[
\frac{1}{1-s_1z-t_1z^2-t_2z^2-\ldots}
\]
with
\[
s_n = q^{n-1}(a + q^{n-1}b + q^n b) \quad \text{and} \quad t_n = q^{3n-1}b(q^n b + a).
\]

From (26) we conclude that \(f(z,a,b,q)\) has a representation as a continued fraction of the form (25) with the same \(s_n, t_n\) together with \(s_0 = a + b\) and \(t_0 = q(a+b)b\).

From this theorem immediately follows.

It is clear that the sequence \((C_n(a,b,q))\) is uniquely determined by these determinants.

**Remark:** It can be shown that the corresponding orthogonal polynomials are
\[
p_n(z,a,b,q) = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} z^k \sum_{j=k}^{n} q^{\binom{n+k+1-j}{2}} \binom{n+j-1}{k} b^{j-k}(a+b)^{n-j}
\]
and
\[
p_n(z,0,1,q) = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \binom{n+k}{2k} z^k.
\]

But we don’t need this result.

c) Let now
\[
h^*(z,a,b,q) := \sum_{k=0}^{n} q^{\binom{k}{2}} r_k(a,b) \frac{(-z)^k}{(1-q)^k}, \] (28)

where \(r_n(a,b,q) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\) is a Rogers-Szegö polynomial, which satisfies
\[
r_n(a,b,q) = (a+b)r_{n-1}(a,b,q) + ab(q^{n-1} - 1)r_{n-2}(a,b,q) \quad \text{(cf. e.g. [4]).}
\]

Therefore we get
\[
h^*(qz,a,b,q) - h^*(z,a,b,q) = -z(a+b)h^*(qz,a,b,q) - qabz^2h^*(q^2z,a,b,q).
\]
If we define
\[ f^*(z,a,b,q) = \frac{h^*(qz,a,b,q)}{h^*(z,a,b,q)}, \]  \hspace{3.5cm} (29)

we see that \( f^*(z,a,b,q) \) satisfies the functional equation
\[ f^*(z,a,b,q) = 1 + (a + b)zf^*(z,a,b,q) + qabz^2 f^*(z,a,b,q)f^*(qz,a,b,q). \]  \hspace{3.5cm} (30)

It is easy to see (compare (17) and (27)) that the series \( f^*(z,a,b) = g(z,a-b,b,1) \) satisfying
\[ f^*(z,a,b) = 1 + (a + b)zf^*(z,a,b) + abz^2 f^*(z,a,b)^2 \]
has the expansion
\[ azf^*(z,a,b) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} N(n,k)a^k b^{n-k} \right) z^n. \]

Therefore we write \( af^*(z,a,b,q) = \sum_{n=0}^{\infty} C_n^*(a,b,q)z^n \)

with
\[ C_n^*(a,b,q) = \sum_{k=1}^{n} N^*(n,k,q)a^k b^{n-k}. \]  \hspace{3.5cm} (31)

This implies \( N^*(n,k,0) = \binom{n-1}{k-1} \).

The first values of \( (N^*(n,k,q))_{k=1}^{n}, n \geq 1 \), are

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 + q & 3 + 2 q + q^2 & 6 + 6 q + 5 q^2 + 2 q^3 + q^4 \\
1 & 4 + 3 q + 2 q^2 + q^3 & 6 + 6 q + 5 q^2 + 2 q^3 + q^4 & 4 + 3 q + 2 q^2 + q^3 + q^4 \\
\end{array}
\]

Computer experiments suggest that \( N^*(n,k,q) \) satisfies the minimal recurrence relation
\[ (E-1)^k (E-q^{\frac{k}{3}}) (E-q^{\frac{2k}{3}}) (E-q^k) \cdots (E-q^{n-1}) N^*(n,k,q) = 0. \]

Since \( f^*(z,a,b) = f^*(z,b,a) \) we see that \( N^*(n,k,q) = N^*(n,n-k+1,q) \).

Let now \( F(z,a,b,q) = 1 + azf^*(z,a,b,q) \). Then by (30)
\[ F(z,a,b,q) = 1 + azF(z,a,b,q) - bzF(qz,a,b,q) + bzF(z,a,b,q)F(qz,a,b,q) \]  \hspace{3.5cm} (32)

If we set \( C_0^*(a,b,q) = 1 \), then (32) implies the recurrence
\[ C_n^*(a,b,q) = aC_{n-1}^*(a,b,q) + b \sum_{k=0}^{n-2} q^k C_k^*(a,b,q) C_{n-1-k}^*(a,b,q). \]  \hspace{3.5cm} (33)

Comparing with [8] (5.5) we see that \( C_4^*(1,s,q) \) are the Pólya-Gessel \( q \)-Catalan numbers, the first values of which are
1,1,1+s,1+2s+qs+s^2,1+3s+2qs+q^2s+3s^2+2qs^2+q^2s^2+s^3,...

By (30) the corresponding \( s_k, t_k \) are given by
\[
s_k = q^{k} (a+b) \quad \text{and} \quad t_k = q^{2k+1} ab.
\]
Therefore their Hankel determinants are
\[
det(C^*_{i+j+1}(a,b,q))_{i,j=0}^n = (ab)^{(n+1) \choose 2} \sum_{q^m} = (ab)^{(n+1) \choose 2} \frac{n(n+1)(2n+1)}{6} \tag{34}
\]
and
\[
det(C^*_{i+j+2}(a,b,q))_{i,j=0}^n = (abq)^{(n+1) \choose 2} \frac{n(n+1)(2n+1)}{6} \frac{a^{n+2} - b^{n+2}}{a-b}. \tag{35}
\]
It is easy to verify that the corresponding orthogonal polynomials are
\[
p_n(z,a,b) = \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{n-k}{2}} \sum_{j=k}^{n} \binom{n+k-j}{k} a^{j-k} b^{n-j}.
\]

In order to compute
\[
det(C^*_{n+j}(a,b,q))_{j=0}^n
\]
we observe that from (32) we get
\[
F(z,a,b,q) = \frac{1-bzF(qz,a,b,q)}{1-bzF(qz,a,b,q)-az} = \frac{1}{\frac{az}{1-aF(z,b,qa,q)} - \frac{az}{1-aF(z,b,qa,q)}} = \frac{1}{\frac{az}{1-aF(z,b,qa,q)}} = \frac{1}{1-azF(z,b,qa,q)}
\]
Therefore we have
\[
F(z,a,b,q) = 1 + azF(z,a,b,q)F(z,b,qa,q)
\]
or
\[
F(z,a,b,q) = 1 + azF(z,a,b,q) + abz^2F(z,a,b,q)F(z,b,qa,q)\nonumber
\]
This gives us \( t_k = q^{2k} ab \) for all \( k \geq 0 \) and therefore we have
\[
det(C^*_{i+j}(a,b,q))_{i,j=0}^n = (ab)^{(n+1) \choose 2} \frac{n(n+1)(n-1)}{3} \quad \tag{36}
\]

In this case the orthogonal polynomials are
\[
p_n(z,a,b) = \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{n-k}{2}} \sum_{j=k}^{n} \binom{n+k-j}{k} a^{j-k} b^{n-j}.
\]
For the special case \((a,b) = (1, s)\) these results have been proved by other methods in [7].
The generating function \( f(z,q) \) of the \( q \)–Catalan numbers \( C_n(q) \) has the continued fraction expansion
\[
f(z,q) = \frac{1}{1-zf(qz,q)} = \frac{1}{1-\frac{z}{1-zf(qz^2,q)}} = \frac{1-qzf(q^2z,q)}{1-qzf(q^2z,q)-z}.
\]
Therefore we get
\[
f(z,q) = 1 + zf(z,q) - qzf(q^2z,q) + qzf(z,q)f(q^2z,q).
\]
This implies \( f(z, q) = F(z, 1, q, q^2) \) and thus the well known (cf. [8]) result that \( C_n(q) = C_n^*(1, q, q^2) \).

This can also directly be seen:

\[
\begin{align*}
\sum_{k=0}^{\infty} q^k \left( \sum_{j=0}^{\infty} \binom{k}{j} q^j \right) \left( \frac{(-z)^k}{(1 + q^2)^k} \right) &= E_z(\frac{-1}{2}) \\
\sum_{k=0}^{\infty} j^k q^j &= (1 + q)^k \; (\text{cf. [4]}).
\end{align*}
\]

From Gauss’s formula \( r_{2n+1}(1, -1) = 0, r_{2n}(1, -1 = (1 - q)(1 - q^3) \cdots (1 - q^{2n-1}) \) (cf e.g.[4]) we conclude in the same way that 

\[
C_2^*(1, -1, q) = (-1)^n q^n C_n(q^2)
\]

and \( C_{2n+2}(1, -1, q) = 0 \).

d) Another interesting special case is given by the \( q \)- Motzkin numbers \( M_n(q) \) which have been considered in [6]. Their generating function \( M(z) = \sum_{n \geq 0} M_n(q) z^n \) satisfies

\[
M(z) = 1 + z M(z) + qz^2 M(z) M(qz).
\]

The first values are

\[
1, 1, q, 1 + 2q + q^2, 1 + 3q + 3q^2 + q^3 + q^4, \cdots
\]

The Hankel determinants are easily seen to be

\[
\det(M_{i+j}(q))^{n}_{i,j=0} = q^{\frac{n(n+1)(2n+1)}{6}}
\]

and

\[
\det(M_{i+j}(q))^{n}_{i,j=0} = q^{\frac{n(n+1)(2n+1)}{6}} d_{n+1} = \frac{1}{2} \binom{n+2}{3} d_{n+1}.
\]

Here \( (d_n)_{n \geq 0} = (1, 1, 0, -1, -1, 0, 1, 1, 0, \cdots) \) is periodic with period 6.

References


