Analysis of Inversions in Spherical and Hyperbolic Geometries

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What is an Inversion?

• In Euclidean geometry:
  – “Reflection” of points across a circle
  – Pts in interior are mapped to the exterior
  – Pts in the exterior are mapped to the interior
  – Pts on the circle are mapped to themselves

• A function, its own inverse
Constructing the Euclidean Inversion

1. We have a point $P$ and a circle of inversion $I$.
2. Take the radius $OR$ from the center $O$ through the point $P$.
3. Construct the chord $AB$ perpendicular to the radius.
4. Construct the tangents to the circle at $A$ and $B$.
5. The point of intersection of the tangents, $P'$ is our required point.

\[ \triangle P'AO \sim \triangle AOP \]

Thus, \[ \frac{OA}{OP'} = \frac{OP}{OA} \]

or, \[ \frac{r}{s'} = \frac{s}{r} \]

or, \[ s' = \frac{r^2}{s} \]
Does this work?

• Consider the unit circle \((r = 1)\). \(s' = \frac{r^2}{s} = \frac{1}{s}\)

• Points inside circle get mapped to the outside
  – If \(s < 1\), \(1/s > 1\)

• Points outside the circle, get mapped to the inside
  – If \(s > 1\), \(1/s < 1\)

• Points on the circle get mapped to themselves
  – If \(s = 1\), \(1/s = 1/1 = 1\) Yes! An inversion!

• The center?
  – If \(s = 0\), \(1/s = \infty\)
Some Examples…

Red: Circle of Inversion

So, shapes are not conserved.
On the other hand...

- Euclidean inversions preserve angles, i.e. they are **conformal**.

The triangle formed between three lines is now a region enclosed by arcs.

But the **angles are preserved**.
Equivalently...

• Let $f: U \rightarrow \mathbb{R}^n$ be a function

• Let $f''(x)$ be the matrix of partial derivatives of $f$

• The function $f$ is conformal if $f''(x)$ is an orthogonal matrix (i.e. its inverse is its own transpose.
  
  – Also if $f''(x)$ is an orthogonal matrix then $\det(f''(x)) = \pm 1$
Euclidean Geometry

- Shortest distance between two points is along a straight line.
- There is a unique straight line through any two given points.
- The angles of a triangle add up to $180^\circ$
Non-Euclidean Geometries: Spherical

• We happen to live on a sphere.
• Shortest distance between two points are along great circles (circles sharing radius with the sphere)
• Through two opposite (antipodal) points, we can find an infinite number of great circles
• Sum of angles in a triangle, add up to more than 180°
Non-Euclidean Geometries: Hyperbolic

- Shortest distance between points is along a hyperbola
- Sides of a triangle are segments of hyperbolas
- Sum of angles on a triangle add up to less than 180°
- We can project down to the Euclidean plane
The Main Idea

• I wanted to know whether we can have inversions in these geometries, and if so, what properties they have.
  – used analogs of the Euclidean construction to see whether we can obtain inversions in other geometries
Spherical Geometry

- Problem!
  - Center of the inversion circle is inverted to the equator.
  - Remember the Euclidean case? (Center inverted to infinity?)
  - We restrict our function to the hemisphere

\[
\arccot \left( \frac{\cot(s)}{\tan s} \right) = \frac{\sin^2 R - \sin^2 s}{\sin^2 R \sin s \cos s} \tan s
\]
Preliminary Findings

\[
R = \pi / 6 \\
R = \pi / 4 \\
R = \pi / 3 \\
R = \pi / 2
\]
Preliminary Findings
Hyperbolic Geometry

\[
\tanh s' = \frac{\tanh^2 R}{\tanh s}
\]

• Problem!
  – There will be a disk inside and concentric to the inversion circle, which does not get inverted.
  – We restrict our function to the rest of hyperbolic space.
Preliminary Findings

\[ R = \frac{\pi}{6} \]

\[ R = \frac{\pi}{4} \]

\[ R = \frac{\pi}{3} \]

\[ R = \frac{\pi}{2} \]
Preliminary Findings
Our Inversions

\[ Spherical: \tan d' = \frac{\tan^2 R}{\tan d} \]

\[ Euclidean : d' = \frac{R^2}{d} \]

\[ Hyperbolic : \tanh d' = \frac{\tanh^2 R}{\tanh d} \]
Further Investigations. Conformal?

\[ Spherical: \tan d' = \frac{\tan^2 R}{\tan d} \]

• Derivative of the function with respect to spherical co-ordinates
  – Determinant = -1 only for \( R = \pi/4 \)
  – For \( R = \pi/4 \), \[
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & -1
  \end{bmatrix}
  \]
    is orthogonal.
  – Inversion on \( R = \pi/4 \) is conformal by our 2\textsuperscript{nd} definition.
Further Investigations. Conformal?

*Hyperbolic: $\tanh d' = \frac{\tanh^2 R}{\tanh d}$*

- Derivative of the function with respect to ‘hyperbolic’ co-ordinates
  - Determinant is never 1 or -1
  - Inconclusive at the moment
Conclusions

• The analog of the Euclidean construction, gives us inversions on certain subsets
• In spherical case, Pi/4 radius seems to give us a conformal function
• Still work to be done!
What next?

• Do the other radii give conformal functions?
  – We know $\pi/4$ in spherical geometry gives us an inversion, what about the rest?

• Project conformally onto the Euclidean plane
  – If we project both pre-image and the image, then we can just determine conformality on the plane.

• How do lines and circles change under these functions?
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