CONSTRUCTION AND ANALYSIS OF INVERSIONS IN $S^2$ AND $H^2$

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ABSTRACT

The construction used to obtain inversions in two-dimensional Euclidean space was modified and applied to obtain inversions on subsets of the two-dimensional spherical and hyperbolic spaces. The functions obtained showed remarkable symmetry with the Euclidean formula, and have similar properties. We investigated the conformality of the functions in the spherical and hyperbolic geometries, but have no conclusive results at this time.

Introduction

We obtain the formulae for inversions as shown below:

- Spherical Geometry: $\tan d' = \frac{\tan^2 R}{\tan d}$
- Euclidean Geometry: $d' = \frac{R^2}{d}$
- Hyperbolic Geometry: $\tanh d' = \frac{\tanh^2 R}{\tanh d}$

where $R$ is the radius of the inverting circle, $d$ is the distance of the original point from the center of the inverting circle, and $d'$ is the distance of the inverted point from the center of the inverting circle. Clearly, these functions are inverses for themselves and show considerable similarity to one another. In addition, we discovered that an inverting circle of radius of $\pi/4$ yields a conformal function in spherical geometry.

Background

a. Inversions

In Euclidean Geometry, an inversion can be considered to be the reflection of points on a circle, such that points in the interior the circle are mapped to the exterior, points in the exterior are
mapped to the interior and points on the circle are mapped to themselves. Inversions are their own inverses, namely, the composition with itself is the identity function.

![Figure 1: Constructing the Euclidean Inversion. (i) Points inside the inverting circle. (ii) Points outside the inverting circle.](image)

**b. Constructing the Euclidean Inversion**

In Euclidean geometry, the inversion can be geometrically constructed (Figure 1). Consider a point P inside a circle I (to be inverted across) with center O. We construct the radius of the circle through P, and then the chord perpendicular to this radius at P. The chord intersects the circle I at the points A and B. The tangents to I at A and B intersect at the point P'.

If the point P to be inverted is outside the circle I, we construct the line segment from the center O to P, and obtain the midpoint M. We construct the circle with center M and radius P. This circle intersects the circle I at the points A and B. The line segment joining A and B intersects OP at the point P'.

**c. Testing the Euclidean Inversion**

Let the radius of the inverting circle be r, OP = d, and OP' = d'.

In the first case, where we invert a point in the interior of the circle of inversion,

\[ \Delta POA \sim \Delta AOP' \]

Thus \[ \frac{OP}{OA} = \frac{OA}{OP'} \]

or \[ \frac{d}{r} = \frac{r}{d'} \]

or \[ d' = \frac{r^2}{d} \]
Similarly, in the second case, where P is in the exterior of I,

\[ \Delta P'O'A \sim \Delta AOP \]

Thus, \[ \frac{OA}{OP'} = \frac{OP}{OA} \]

or, \[ \frac{r}{d''} = \frac{d}{r} \]

or, \[ d'' = \frac{r^2}{d} \]

Notably, \textit{when } d < r, \textit{then } d' > r, \textit{when } d > r, \textit{then } d' < r, \textit{and when } d = r, d' = r. \textit{Thus,} this satisfies the conditions of being an inversion.

d. Examples

![Figure 2: Objects being inverted across a circle, on the Euclidean plane. (i) The green circle is inverted to the blue circle. (ii) The green line is inverted to the blue circle.](image)

In the first example, the portion of the green circle inside the inversion circle is mapped to the portion of the blue circle in the exterior of the inversion circle. The part of the green circle outside the inversion circle is mapped to the part of the blue circle in the interior of the inversion circle. It is clear that distances are not preserved. The blue and green curves intersect on the inversion circle, since points of the green circle on the inversion circle get mapped to themselves, by the characteristic property of inversions.

In the second example, the green line is mapped to the blue circle. Again, the two curves intersect on the inversion circle. Similarly, inside of green corresponds to outside of blue, vice versa. Certainly, shapes are not preserved under Euclidean inversions. In both these examples, it serves to keep in mind that the blue circles would also get inverted to the green circle and line respectively, due to the inversive nature of the function.
e. Conformality

Figure 3: The angles before and after inversion are equal.

Despite not conserving distances, inversions do preserve angles between curves, i.e. they are conformal. We know that if \( f: \mathbb{X} \to \mathbb{R}^n \) is a function and \( f'(x) \) is the matrix of partial derivatives, \( f \) is conformal if \( f'(x) \) is an orthogonal matrix. A matrix orthogonal if its transpose is its inverse, therefore, the determinant of an orthogonal matrix is always \( \pm 1 \). This gives us a method of determining whether functions are conformal.

f. A survey of 2-dimensional Euclidean and Non-Euclidean geometries.
   i. Euclidean Geometry

In 2-dimensional Euclidean geometry, or \( \mathbb{E}^2 \), the shortest distance between two points is along a straight line. There is a unique straight line between any two given points, and the angles of a triangle add up to 180°. The inversions discussed so far are all in \( \mathbb{E}^2 \).

   ii. Spherical Geometry

Figure 3: A triangle on a sphere with angle sum 270°

The standard model for 2-dimensional spherical geometry is the unit sphere \( S^2 \) in \( \mathbb{R}^3 \). The spherical distance between two points \( x \) and \( y \) is given by the Euclidean angle between their position vectors. On the unit sphere, therefore, this is equal to the length of the arc of the
unique great circle (unit circles lying on the sphere) joining \( x \) and \( y \). Spherical geometry is one of the two homogeneous examples of non-Euclidean geometries, since Euclid’s fifth postulate does not hold true on a sphere. Great circles are the geodesics on the sphere and it is clear that through two antipodal points there exist an infinite number of great circles. Also, the angle sum of triangles is always greater than 180° (Figure 3).

iii. **Hyperbolic Geometry**

![Figure 4: A triangle on \( H^2 \)](image)

2-dimensional hyperbolic geometry (\( H^2 \)) can be considered to be the geometry on the upper sheet of a unit hyperboloid in \( \mathbb{R}^3 \). Geodesics on the hyperboloid are unsurprisingly hyperbolas on its surface, and the distance between two points may be considered to be the arc length of the unique hyperbolic segment between them. In \( H^2 \), the angle sum of a triangle is always less than 180°. Hyperbolic geometry is also a non-Euclidean geometry.

Spherical and hyperbolic geometries show an interesting duality, which begins with the nature of how Euclid’s fifth postulate is negated in each case. Euclid’s fifth postulate is equivalent to the modern parallel postulate, which states that through a point outside a given straight line there is exactly one straight line parallel to the given line. In spherical geometry, where great circles may be considered to be straight lines, through a point outside a given great circle, there are no great circles parallel to it (all great circles must necessarily intersect one another). In hyperbolic geometry, where hyperbolas may be considered to be straight lines, through a point outside a given hyperbola, there are infinitely many hyperbolas parallel to it. Notably, Euclidean geometry is frequently midway between spherical and hyperbolic geometries. For instance, the curvature of \( S^2 \) is 1, that of the Euclidean plane is 0, and that of \( H^2 \) is -1.
Modifying the Construction for Spherical and Hyperbolic Geometries

a. Spherical Geometry

In the spherical case, straight lines are replaced by great circles. The inverting circle (given in Figure 5 by ZYZ’) is some circle on the surface of the sphere, not necessarily a great circle. The center of the inverting circle is given by P, and we shall invert the point X.

As in the Euclidean case, our first step is to take the radial “line” (in this case the great circle PXY) through X. We take the great circle perpendicular to PXY through X. This great circle intersects the inverting circle at Z and Z’. We take the radii to Z and Z’ and construct the great circles perpendicular to them. These tangent great circles intersect at X’.

Let \( d_5(P,Y) = R; \)
\( d_5(P,Z) = R; \)
\( d_5(P,X) = d; \)
\( d_5(X,Z) = b; \)
\( d_5(P,X') = d'; \)
\( \angle PX'Z = q; \)

\( \angle XPZ = B; \)
\( \angle PZX = D; \)

Applying the spherical laws of sines and cosines,

\[
\text{From } \triangle XX'Z, \quad \frac{\sin(\theta' - d)}{\sin\left(\frac{\pi}{2} - D\right)} = \frac{\sin b}{\sin \theta}
\]

\[
\text{From } \triangle PX'Z, \quad \frac{\sin \theta'}{\sin \left(\frac{\pi}{2} - D\right)} = \frac{\sin R}{\sin \theta} \Rightarrow \sin \theta = \frac{\sin R}{\sin \theta'}
\]
From $\triangle PXZ$, \[
\frac{\sin d}{\sin D} = \frac{\sin R}{\sin \frac{\pi}{2}} \Rightarrow \sin D = \frac{\sin d}{\sin R}
\]

Also, \[
\frac{\cos \frac{\pi}{2} \cos D + \cos B}{2} = \frac{\cos B}{\sin D}
\]

Also, \[
\frac{\sin b}{\sin B} = \frac{\sin R}{\sin \frac{\pi}{2}} \Rightarrow \sin B = \frac{\sin b}{\sin R}
\]

\[
\sin b = \sqrt{1 - \cos^2 b} = \sqrt{1 - \frac{\cos^2 B}{\sin^2 D}} = \frac{1}{\sin D} \sqrt{\sin^2 D - \cos^2 B}
\]

\[
= \frac{1}{\sin D} \sqrt{\sin^2 d - 1 + \sin^2 b \sin R} \Rightarrow \sin^2 b \sin^2 D \sin^2 R = \sin^2 d - \sin^2 R + \sin^2 b
\]

or, \[
\sin^2 b = \frac{\sin^2 d - \sin^2 R}{\sin R \sin^2 D - 1} = \frac{\sin^2 R - \sin^2 d}{\cos^2 d}
\]

\[
\frac{\sin(b'-d)}{\cos D} = \frac{\sin b}{\sin D} \frac{\sin b' \sin d'}{\sin R}
\]

or, \[
\frac{\sin d \cos d - \cos d \sin d'}{\cos D} = \frac{\sin d' \sqrt{\sin^2 R - \sin^2 d}}{\sin R \cos d}
\]

or, \[
\cot d - \cot d' = \frac{\sqrt{1 - \sin^2 D} \sqrt{\sin^2 R - \sin^2 d}}{\sin R \cos d \sin d'}
\]

or, \[
\cot d' = \cot d - \frac{\sin^2 R - \sin^2 d}{\sin R \cos d \sin d'} = \frac{\cos^2 d \sin^2 R - \sin^2 R + \sin^2 d}{\sin^2 R \cos d \sin d'} = \frac{\sin^2 d \cos R}{\sin^2 R \cos d \sin d'} = \cot^2 R \tan d
\]

or, \[
\tan d' = \frac{\tan^2 R}{\tan d}
\]
Notice, the certainly this function maintains the necessary inversive nature, namely the composition of the function with itself is the identity function. Also, it is not possible to invert beyond the hemisphere, therefore, we restrict the function to the hemisphere. On the other hand, it is clear that points in the interior of the inverting circle get mapped to the exterior and vice versa, and points on the circle are mapped to themselves. Therefore, on the restricted domain, this function is indeed an inversion.

b. Hyperbolic Geometry

![Figure 6: Constructing the inversion on H². Points on H² are projected down to the plane.]

We start with projecting points in H² perpendicularly down to the plane. The inverting circle (shown in red in Figure 6) has center P, and X is the point to be inverted. We take the radius through X and the hyperbola perpendicular to this radius at the point X. This chord hyperbola intersects the inverting circle at Z and Z’. Tangent hyperbolas at these points intersect at X’. Let $d_H(P,Y)=R$; $d_H(X,Z)=b$; $d_H(P,Z)=R$; $d_H(P,X)=d$; $d_H(P,X')=d'$; $\angle PX'Z=q$; $\angle XPZ=B$; $\angle PZX=D$.

Applying the hyperbolic laws of sines and cosines,

From $\triangle XX'Z$, \[ \frac{\sinh(d'-d)}{\sin\left(\frac{\pi}{2} - D\right)} = \frac{\sin b}{\sin \theta} \]

From $\triangle PX'Z$, \[ \frac{\sinh d'}{\sin \frac{\pi}{2}} = \frac{\sin R}{\sin \theta} \Rightarrow \sin \theta = \frac{\sin R}{\sinh d'} \]

\[ \text{Figure 6: Constructing the inversion on } H^2. \text{ Points on } H^2 \text{ are projected down to the plane.} \]
Also, \[
\sinh b = \frac{\sinh R}{\sin \frac{\pi}{2}} \Rightarrow \sin B = \frac{\sinh b}{\sinh R}
\]

From \(\triangle PXZ\), \[
\frac{\sinh d}{\sin D} = \frac{\sinh R}{\sin \frac{\pi}{2}} \Rightarrow \sin D = \frac{\sinh d}{\sinh R}
\]

Also, \[
\cos \frac{\pi}{2} \cos D + \cos B = \frac{\cos B}{\sin D}
\]

\[
\sinh b = \sqrt{\cosh^2 b - 1} = \sqrt{\frac{\cosh^2 B}{\sin D} - 1} = \frac{1}{\sin D} \sqrt{\cosh^2 B - \sin^2 D}
\]

\[
= \frac{1}{\sin D} \sqrt{1 - \frac{\sinh^2 b}{\sinh^2 R} - \frac{\sinh^2 d}{\sinh^2 R}}
\]

\[
\Rightarrow \sinh b \sinh^2 D \sinh R = \sinh R - \sinh b - \sinh d
\]

or, \[
\sinh b = \frac{\sinh R - \sinh d}{\sinh \sinh^2 D + 1} = \frac{\sinh R - \sinh d}{\cosh^2 d}
\]

\[
\frac{\sinh (b''-d)}{\cos D} = \frac{\sinh b}{\sin D} = \frac{\sinh b \sinh d'}{\sinh R}
\]

or, \[
\frac{\sinh d \cosh d' - \cosh d \sinh d'}{\cos D \sinh d} = \frac{\sinh d'}{\sinh R \cos d' \sinh d}
\]

or, \[
\coth d - \coth d' = \frac{\sqrt{1 - \sin^2 D \sinh R - \sin^2 d'}}{\sinh R \cos d' \sinh d}
\]

or, \[
\coth d' = \coth d - \frac{\sinh^2 R - \sinh^2 d}{\sinh R \cos d' \sinh d}
\]

\[
\frac{\cosh d' \sinh R - \sinh R \cosh d + \sinh^2 d}{\sinh R \cos d' \sinh d}
\]

\[
= \frac{\sinh d \cosh R}{\sinh R \cos d' \sinh d}
\]

\[
= \coth^2 R \tan d'
\]

or, \[
\tan d' = \frac{\tanh^2 R}{\tan d'}
\]
We can see, once again, the inversive nature of this function. However, this function as it stands is not well-defined. Namely, a disc concentric to the inverting circle does not get mapped anywhere. The radius of this disc can be found as follows:

\[
\tanh^{-1}\left(\frac{\tanh^2 R}{\tanh d}\right) = \infty
\]
\[
\therefore \left(\frac{\tanh^2 R}{\tanh d}\right) = 1
\]

or, \(d = \tanh^{-1}(\tanh^2 R)\)

Notably, the radius of this circle increases with increase in the radius of the inverting circle. We define the inversion on the remainder of \(H^2\). On the other hand, it is clear that points in the interior of the inverting circle get mapped to the exterior and vice versa, and points on the circle are mapped to themselves. Therefore, on the restricted domain this function is an inversion.

**Results**

a. **Inversions**

The symmetry of the obtained graphs (Figure 6 and 7) around \(y=x\) verifies that these functions are their own inverses. The second set of graphs show that there are discs around the center of the inverting circle which do not get mapped under the hyperbolic inversion.

b. **Investigation of conformality**

The matrix of partial derivatives with respect to spherical co-ordinates for the inversion on the hemisphere is given below:

If \(x = (r, \theta, d)\)

then \(f(x) = (r, \theta, \cot^{-1}\left(\frac{\cot^2 R}{\cot d}\right))\)
\[ f'(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-\cot^2 R(1 + \cot^2 d)}{(1 + \frac{\cot^4 R}{\cot^2 d})\cot^2 d} \end{bmatrix} \]

Since, \( \det(f'(x)) = 1 \) for all \( d \) if and only if \( R = \frac{\pi}{4} \), we can infer that the inversion over a circle of radius \( \frac{\pi}{4} \) is conformal on the hemisphere.

Figure 7: Graphs for inverting on the hemisphere. The \( x \) and \( y \) axes represent the distances of the original and inverted point respectively from the center of the inverting circle.
Future Directions

We would like to investigate the behavior of angles under these inversions in greater detail. A method we have considered using involves taking conformal projections of curves before and after inversion, onto the Euclidean plane. If angles are preserved on the Euclidean plane we would be able to infer conformality on the original surface. We would also like to determine how lines and circles are changed under the inversions.

Figure 7: Graphs for inverting on the hemisphere. The x and y axes represent the distances of the original and inverted point respectively from the center of the inverting circle.