Satellite operations and fractal structures on knot concordance

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Satellite operations on knots

Figure: The satellite operation on knots
Any knot $P$ in a solid torus gives a function on the set of knots.

\[
P : \mathcal{K} \rightarrow \mathcal{K}
\]

\[
K \mapsto P(K)
\]
Knot concordance

Definition

Knots $K_0$, $K_1$ are concordant if they cobound a smoothly embedded annulus in $S^3 \times [0, 1]$. Knots modulo concordance form the knot concordance group $C$.

A knot is slice if it is concordant to the unknot.
Topological knot concordance

Definition

Knots \( K_0, K_1 \) are \textit{topologically concordant} if they cobound a locally flat, topologically embedded annulus in \( S^3 \times [0, 1] \). Knots modulo topological concordance form the \textit{topological knot concordance group} \( C_{\text{top}} \).

A knot is \textit{topologically slice} if it is topologically concordant to the unknot.
Exotic knot concordance

Definition

Knots \( K_0, K_1 \) are *exotically concordant* if they cobound a smoothly embedded annulus in a smooth manifold \( M \) homeomorphic to \( S^3 \times [0, 1] \), i.e. a possibly exotic \( S^3 \times [0, 1] \). Knots modulo exotic concordance form the *exotic knot concordance group* \( C_{\text{ex}} \).

If the smooth 4–dimensional Poincaré Conjecture holds, then \( C = C_{\text{ex}} \).

A knot is *exotically slice* if it is exotically concordant to the unknot.
Satellite operators on knot concordance

Any knot in a solid torus gives a well-defined map on knot concordance classes, called a satellite operator. That is, we have the following commutative diagram.

\[
\begin{array}{ccc}
K & \xrightarrow{P} & K \\
\downarrow & & \downarrow \\
C_* & \xrightarrow{P} & C_ *
\end{array}
\]

for any \( * \in \{\emptyset, \text{top}, \text{ex}\} \).
How do satellite operators act on knot concordance?

Figure: The untwisted Whitehead double of a knot $K$
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Long-standing conjecture: $\text{Wh}(K)$ slice $\Rightarrow K$ slice.
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Figure: The untwisted Whitehead double of a knot $K$

Long-standing conjecture: $\text{Wh}(K)$ slice $\Rightarrow K$ slice.
This can be restated as: what is the ‘kernel’ of $\text{Wh} : C \rightarrow C$?
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1. is \( P \) ‘weakly injective’? That is, if \( P(K) = 0 \), is \( K = 0 \)?
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2. is $P$ injective? That is, if $P(K) = P(J)$, is $K = J$?
3. does $P$ preserve linear independence? That is, if $\{K_i\}$ is linearly independent, is $\{P(K_i)\}$?
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5. is $P$ surjective?
6. what are the ‘dynamics’?
7. any other question you might ask about functions.
Connected-sum is a satellite operation.

![Diagram](image)

**Figure:** The pattern for connected-sum with the knot $K$

Connected-sum is both injective and surjective on any $C_\ast$. 
Previous results

Hedden (2007): if $\tau(K) > 0$, then $Wh^i(K)$ is not slice for any $i \geq 0$.

Cochran–Harvey–Leidy (2011): large classes of ‘robust doubling operators’ (winding number zero) injectively map large infinite subgroup of $C$ to an independent set.

Hedden–Kirk (2012): the Whitehead doubling operator preserves the linear independence of an infinite independent set of torus knots. (later generalized by Juanita Pinzón-Caicedo)
Injectivity of satellite operators

Theorem (Cochran–Davis–R.)

Any ‘strong winding number ±1’ satellite operator is injective on $C_{top}$ and $C_{ex}$.

Thus, modulo smooth 4DPC, any strong winding number ±1 satellite operator is injective on $C$. 

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Thus, modulo smooth 4DPC, any strong winding number ±1 satellite operator is injective on \( C \).

Corollary: if \( \tau(K) \neq 0 \), then \( P^i(K) \) is not slice for any winding number ±1 satellite operator \( P \) with \( P(U) \) slice, for any \( i \geq 0 \).

(There are analogous results for other non-zero winding numbers \( w \), in terms of concordance in \( \mathbb{Z}[\frac{1}{w}] \)-homology \( S^3 \times [0, 1] \); in particular, any winding number ±1 satellite operator is injective on concordance classes in integral homology \( S^3 \times [0, 1] \). For brevity, we will not discuss this much more.)
Strong winding number $\pm 1$

A pattern $P$ is ‘strong winding number $\pm 1$’ if the meridian of the solid torus normally generates $\pi_1(S^3 - P(U))$.

cf. $P$ is winding number $\pm 1$ if the meridian of the solid torus generates $H_1(S^3 - P(U))$.

Figure: The Mazur pattern
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If $P(U)$ is unknotted, strong winding number $\pm 1$ is the same as winding number $\pm 1$. 

Figure: The Mazur pattern
Proof of injectivity

First we prove weak injectivity for slice patterns.

Recall that a knot $K$ is (topologically or exotically) slice if and only if the zero surgery $M_K$ bounds a 4–manifold $W$ where $W$ is a homology circle and the meridian of $K$ normally generates $\pi_1(W)$. 
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Lemma: If $R$ is strong winding number $\pm 1$ with $R(U)$ (topologically or exotically) slice then $M_{R(K)}$ is homology cobordant to $M_K$ via a 4–manifold $V$ where $\pi_1(V)$ is normally generated by the meridian of $K$. 
Proof of injectivity

Now suppose that $R(K)$ is slice, $R(U)$ is slice, and $R$ is strong winding number $\pm 1$. 

\[ W \]

\[ M_{R(K)} \]
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Proof of injectivity

Now suppose that $R(K)$ is slice, $R(U)$ is slice, and $R$ is strong winding number $\pm 1$.

By the previous lemma, $K$ is slice, and thus slice strong winding number $\pm 1$ satellite operators are weakly injective.
Proof of injectivity

Now, suppose $P(K) = P(J)$ (i.e. concordant in the relevant category), where $P$ is strong winding number $\pm 1$ (not necessarily slice).
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Since $K# - K$ is slice, $J = K# - K#J$, and thus,

$$P(J) = P(K# - K#J)$$
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and then,

$$-P(K)\# [P(K\# - K\#J)] = 0$$
Proof of injectivity

We know that $-P(K)\# [P(K\#(-K\#J))]$ is slice. This knot is shown below.

\[\begin{tikzpicture}
  \node (P) at (0,0) {$P$};
  \node (P') at (3,0) {$P$};
  \node (K) at (1.5,-1.5) {$K$};
  \node (K') at (2.5,-1.5) {$-K\#J$};
  \draw (P) -- (K);
  \draw (P') -- (K');
  \draw (K) -- (K');
\end{tikzpicture}\]
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Note that this is a satellite with a ribbon pattern and companion $-K\#J$. The pattern is strong winding number one.
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Note that this is a satellite with a ribbon pattern and companion $-K \# J$. The pattern is strong winding number one.

Thus, by weak injectivity for satellite operators with slice patterns, $-K \# J$ is slice, and thus $K = J$. 
Satellite operators form a monoid

Proposition

The satellite operation gives a monoid action on knots, i.e.

$$(P \ast Q)(K) = P(Q(K))$$
Patterns and homology cylinders

Given a pattern $P$ in a solid torus $ST$, let $E(P)$ denote the complement $ST - P$.

$E(P)$ is a 3–manifold with two toral boundary components, specifically a homology cylinder.
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Homology cylinders, modulo homology cobordism, form a group under stacking (J. Levine).

Let $\hat{S}_*$ be the group of the ‘strong’ homology cylinders under ‘strong’ homology cobordism.
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Let $\hat{S}_*$ be the group of the ‘strong’ homology cylinders under ‘strong’ homology cobordism.

There is a monoid homomorphism from the monoid of strong winding number $\pm 1$ patterns to the group $\hat{S}_*$. 

Arunima Ray (Brandeis)

Satellite operations and fractals

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Homology cylinders act on knots in homology 3–spheres

Let \( V \) be a homology cylinder. Given a knot \( K \) in a homology 3–sphere \( Y \), carve out \( N(K) \), a solid torus neighborhood of \( K \).
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$$Y - N(K) \quad V$$

$$\partial N(K) = \partial_- V \quad \partial_+ V$$
Let $V$ be a homology cylinder. Given a knot $K$ in a homology 3–sphere $Y$, carve out $N(K)$, a solid torus neighborhood of $K$.

We obtain a 3–manifold with a single torus boundary component. We can canonically glue in a solid torus to get a homology 3–sphere. The core of this solid torus is the new knot.
Generalizations of knot concordance

Let $\widehat{C}_*$ be the group of knots in homology spheres modulo concordance in ‘strong’ homology cobordisms.
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There are injective homomorphisms $C_* \hookrightarrow \hat{C}_*$.

(Davis–R.): $\hat{S}_*$ acts on $\hat{C}_*$ by a group action.
Theorem (Davis–R.)

For \( * = \text{ex or top}, \) and any strong winding number one satellite operator \( P, \) the following diagram commutes.

\[
\begin{array}{c}
\mathbb{C}_* \xrightarrow{P} \mathbb{C}_* \\
\downarrow \quad \downarrow \\
\hat{\mathbb{C}}_* \xrightarrow{E(P)} \hat{\mathbb{C}}_*
\end{array}
\]

Since \( \hat{S}_* \) gives a group action on \( \hat{\mathbb{C}}_* \), each \( E(P) \in \hat{S}_* \) acts via a bijection. The Cochran–Davis–R. injectivity result for strong winding number \( \pm 1 \) satellite operators follows.
Satellite operators as group actions

Thus, the classical satellite operation on $C_\ast$ is a restriction of a group action on $\hat{C}_\ast$. 
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Since $E(P)$ is an element of a group, it has an inverse $E(P)^{-1}$.
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Since $E(P)$ is an element of a group, it has an inverse $E(P)^{-1}$.

$P$ is surjective on $C_\ast$ if and only if $E(P)^{-1}(C_\ast) \subseteq C_\ast$. 
Satellite operators as group actions

Thus, the classical satellite operation on $\mathcal{C}_*$ is a restriction of a group action on $\hat{\mathcal{C}}_*$.

Since $E(P)$ is an element of a group, it has an inverse $E(P)^{-1}$.

$P$ is surjective on $\mathcal{C}_*$ if and only if $E(P)^{-1}(\mathcal{C}_*) \subseteq C_*$. 

**Theorem (Davis–R.)**

Let $P \subseteq ST = S^1 \times D^2$ be winding number one. If the meridian of $P$ is in the normal subgroup of $\pi_1(E(P))$ generated by the meridian of $ST$, then $P$ is strong winding number one and there exists a strong winding number one pattern $\overline{P}$ such that $E(\overline{P}) = E(P)^{-1}$ as homology cylinders.

In particular, $\overline{P}(P(K))$ is (exotically or topologically) concordant to $K$ for any knot $K$. 
Satellite operators as group actions

Thus, the classical satellite operation on $C_*$ is a restriction of a group action on $\hat{C}_*$.

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In particular, $\overline{P}(P(K))$ is (exotically or topologically) concordant to $K$ for any knot $K$.

Consequently, $P : C_* \rightarrow C_*$ is a bijection.
For each \( m \geq 0 \), the satellite operator \( P_m \) shown below has an inverse satellite operator \( \overline{P_m} \) which can be explicitly drawn, i.e. \( \overline{P_m}(P_m(K)) \) is concordant to \( K \) for any knot \( K \). Moreover, each \( P_m : \mathcal{C}_* \rightarrow \mathcal{C}_* \) is bijective and \( P_m \) is distinct from all connected-sum operators in \( \hat{S}_* \).

Note that it is still possible that, for some fixed knot \( J \), \( P_m(K) = J \# K \) for all \( K \), i.e. it is not known whether patterns act faithfully.
In contrast, recall from yesterday that the Mazur satellite operator is non-surjective on \( \mathcal{C} \) (A. Levine).

In particular, Levine showed that no knot \( J \) with \( \varepsilon(J) = -1 \) is in the image of the Mazur satellite operator.

Note that it is not known whether the Mazur satellite operator is the identity function on \( C_{\text{top}} \).

**Figure:** The Mazur pattern
K. Park: $Wh(T_{2,2m+1})$ and $Wh^2(T_{2,2m+1})$ generate a $\mathbb{Z} \oplus \mathbb{Z}$ summand of the subgroup of topologically slice knots in $C$.

R. : For several classes of strong winding number $\pm 1$ patterns $P$ (including the Mazur pattern) and infinitely many knots $K$, $P^i(K) \neq P^j(K)$ in $C_{ex}$ for any $i \neq j \geq 0$.
(For the Mazur pattern, this can be improved by A. Levine’s computation of $\tau$–invariants.)

Feller–J. Park–R. : Let $M$ be the Mazur satellite operator. There exists an infinite family of topologically slice knots $\{K_i\}$ such that for all $r \geq 0$, $\{M^r(K_i)\}$ generates a subgroup of $C$ of infinite rank.
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Conjecture (Cochran–Harvey–Leidy, 2011)

The knot concordance group $C$ is a fractal.
The knot concordance group has fractal properties

Figure: The Mazur pattern $M$

Cochran–Davis–R. : $M$ is injective on $C_{ex}$ and $C_{top}$.

A. Levine: $M$ is not surjective on $C$. Moreover,

$$Im(M) \supsetneq Im(M^2) \supsetneq Im(M^3) \supsetneq \cdots$$

What about scale?
The knot concordance group has fractal properties

To properly address the question of scale we need some notion of distance on $C_*$. This was started by Cochran–Harvey, with further work by Cochran–Harvey–Powell (see talk on Saturday).