Satellite operations on knots, and fractals

Arunima Ray
Rice University

STEM Colloquium, University of Wisconsin–Eau Claire

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Take a piece of string, tie a knot in it, glue the two ends together.
Take a piece of string, tie a knot in it, glue the two ends together. A knot is a closed curve in space which does not intersect itself anywhere.
Two knots are **equivalent** if we can get from one to the other by a continuous deformation, i.e. without having to cut the piece of string.

Figure: All of these pictures are of the same knot, the *unknot* or the *trivial knot*.
‘Adding’ two knots

![Knot Diagrams](image)

**Figure**: The connected sum operation on knots

The (class of the) unknot is the identity element, i.e. $K \# \text{Unknot} = K$
‘Adding’ two knots

![Image of knots](image)

Figure: The connected sum operation on knots

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However, there are no inverses for this operation. In particular, if neither $K$ nor $J$ is the unknot, then $K \# J$ cannot be the unknot either.

(In fact, we can show that $K \# J$ is more complex than $K$ and $J$ in a precise way.)
A 4–dimensional notion of a knot being ‘trivial’

A knot $K$ is equivalent to the unknot if and only if it is the boundary of a disk.
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We want to extend this notion to four dimensions.
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Figure: Schematic picture of the unknot
A 4–dimensional notion of a knot being ‘trivial’

**Definition**

A knot $K$ is called **slice** if it bounds a disk in four dimensions as above.

**Figure** : Schematic pictures of the unknot and a slice knot
**Definition**

Two knots $K$ and $J$ are said to be **concordant** if they cobound a smooth annulus in $\mathbb{R}^3 \times [0, 1]$. 
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\[ \mathbb{R}^3 \times [0, 1] \]
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The knot concordance group

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This means that for every knot $K$ there is some $-K$, such that $K \# -K$ is a slice knot.

We call the group of knot concordance classes the knot concordance group and denote it by $C$. 
Goal

Goal: study the knot concordance group $C$ by studying functions on it.

In particular, this will show that $C$ has the structure of a fractal.
Fractals are objects that exhibit ‘self-similarity’ at arbitrarily small scales.

i.e. there exist families of ‘injective’ functions from the set to smaller and smaller subsets.
The satellite operation is a generalization of the connected sum operation. Here $P$ is called a satellite operator, and $P(K)$ is called a satellite knot.
Any knot $P$ in a solid torus gives a function on the set of all knots

$$P : \mathcal{K} \to \mathcal{K}$$

$$K \to P(K)$$

These functions descend to give well-defined functions on the knot concordance group.

$$P : \mathcal{C} \to \mathcal{C}$$

$$K \to P(K)$$
The knot concordance group has fractal properties

Recall that a fractal is a set which admits self-similarities at arbitrarily small scales, i.e. there exist infinitely many injective functions from the set to smaller and smaller subsets.

Theorem (Cochran–Davis–R., 2012)

For large (infinite) classes of satellite operators $P$, $P : C \rightarrow C$ is injective (modulo the smooth 4–dimensional Poincaré Conjecture).

Theorem (R., 2013)

There exist infinitely many satellite operators $P$ and a large class of knots $K$ such that $P^i(K) \neq P^j(K)$ for all $i \neq j$. 
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2. There is a four-dimensional equivalence relation on knots, called ‘concordance’, which gives the set of knots a group structure.
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2. There is a four-dimensional equivalence relation on knots, called ‘concordance’, which gives the set of knots a group structure.

3. By studying the action of ‘satellite operators’ on knots, we can see that the knot concordance group has fractal properties.