Shake slice and shake concordant knots

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Representing homology classes

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For a 4–manifold $X$, given $\alpha \in H_2(X; \mathbb{Z})$, can we represent $\alpha$ by an embedded sphere?
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*For a 4–manifold $X$, given $\alpha \in H_2(X; \mathbb{Z})$, can we represent $\alpha$ by an embedded sphere?*

This is related to the minimal genus question, i.e. given $\alpha$, what is the minimal genus of a surface representative of $\alpha$?
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**Definition (Akubulut)**

The knot \( K \) is said to be shake slice if the generator of \( H_2(V_K) \) can be represented by an embedded sphere.
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Not all knots are shake slice (Akbulut).
Shake slice knots

Figure: A shaking of the knot $K$

Proposition (Cochran–R.)
A knot $K$ is shake slice if and only if some shaking of $K$ bounds a genus zero surface in $B^4$. 

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January 6, 2016 4 / 11
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Definition

Two knots $K$ and $J$ are said to be **concordant** if they cobound an annulus in $S^3 \times [0, 1]$. 
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Question
Are there knots that are shake concordant but not concordant?
For any knot \( K \), let \( M_K \) denote the manifold obtained by performing 0–framed surgery on \( S^3 \) along \( K \).
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Cochran–Franklin–Hedden–Horn
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Results

Theorem (Cochran–R.)

There exist infinitely many (topologically slice) knots that are distinct in concordance but are pairwise shake concordant.

In addition, $\tau$, $s$, and slice genus all fail to be invariants of shake concordance.
The previous result follows from a characterization theorem for shake concordant knots.

**Theorem (Cochran–R.)**

\[ K \text{ is shake concordant to } J \text{ if and only if there exist winding number one patterns } P, Q, \text{ with } P(U), Q(U) \text{ slice such that } P(K) \text{ is concordant to } Q(J). \]
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**Figure:** The satellite operation on knots
Corollary (Cochran–R.)

The equivalence relation on the set of isotopy classes of knots generated by shake concordance is the same as the one generated by concordance and setting a knot equal to its satellites under slice winding number one patterns.
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We also get a characterization of shake slice knots.

Corollary (Cochran–R.)

$K$ is shake slice if and only if there exists a winding number one pattern $P$ such that $P(U)$ and $P(K)$ are slice.

This follows from the characterization theorem, since a knot is shake slice if and only if it is shake concordant to the unknot.