Knots, four dimensions, and fractals

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Examples of knots

Do you know how to make these? No? WELL YOUR POCKET DOES.
Examples of knots

FORTY KNOTS
A VISUAL AID FOR KNOT TYING
OFFICIAL EQUIPMENT—BOY SCOUTS OF AMERICA

The Scout Seal is Your Guarantee of Quality, Excellence and Performance

OVERHAND KNOT
SAILOR'S KNOT
SQUARE KNOT
LARK'S HEAD
FIGURE EIGHT KNOT

STEVEDORE'S KNOT
KILLLICK HITCH
SHEET BEND
SHEET BEND DOUBLE
TIMBER HITCH
LARIAT LOOP

OVERHAND BOW
CAT'S PAW
CLOVE HITCH
BLACKWALL HITCH
GRANNY KNOT

FISHERMAN'S KNOT

DOUBLE CARRICK BEND
Mathematical knots

Take a piece of string, tie a knot in it, glue the two ends together.

Definition

A (mathematical) knot is a closed curve in space with no self-intersections.
Why knots?

Knot theory is a subset of the field of topology.

**Theorem (Lickorish–Wallace, 1960s)**

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Modern knot theory has applications to algebraic geometry, statistical mechanics, DNA topology, quantum computing, . . . .
# Big questions in knot theory

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2. How can we tell if two knots are distinct?

3. Can we quantify the ‘knottedness’ of a knot?
Genus of a knot

Proposition (Frankl–Pontrjagin, Seifert, 1930s)

Any knot bounds a surface in $\mathbb{R}^3$. 
**Fundamental theorem in topology**

*Surfaces are classified by their genus.*

- genus = 0
- genus = 1
- genus = 2
Genus of a knot

Fundamental theorem in topology

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Definition

The *genus* of a knot $K$, denoted $g(K)$, is the least genus of surfaces bounded by $K$. 

\[
\begin{align*}
\text{genus} &= 0 \\
\text{genus} &= 1 \\
\text{genus} &= 2
\end{align*}
\]
Proposition

If $K$ and $J$ are equivalent knots, then $g(K) = g(J)$. 

If $T$ is the trefoil knot, $g(T) = 1$. Therefore, the trefoil is not equivalent to the unknot.
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**Connected sum of knots**

*Figure:* The connected sum of two trefoil knots, $T \# T$
Connected sum of knots

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Proposition

*Given two knots $K$ and $J$, $g(K\#J) = g(K) + g(J)$.*
Connected sum of knots

![Diagram of connected sum of knots](image)

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**Corollary:** There exist infinitely many distinct knots!
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*Corollary:* We can never add together non-trivial knots to get a trivial knot.
Slice knots

Recall that a knot is equivalent to the unknot if and only if it is the boundary of a disk in $\mathbb{R}^3$.

**Definition**

A knot $K$ is *slice* if it is the boundary of a disk in $\mathbb{R}^3 \times [0, \infty)$. 

*Figure: Schematic picture of the unknot*
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Figure: Schematic picture of the unknot and a slice knot
Examples of slice knots
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Knots of this form are called ribbon knots. Knots, modulo slice knots, form a group called the knot concordance group, denoted $C$. 
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Conjecture (Cochran–Harvey–Leidy, 2011)

*The knot concordance group $C$ is a fractal.*
Satellite operations on knots

**Figure:** The satellite operation on knots
Any knot $P$ in a solid torus gives a function on the knot concordance group,

$$P : \mathcal{C} \rightarrow \mathcal{C}$$

$$K \mapsto P(K)$$

These functions are called satellite operators.
The knot concordance group has fractal properties

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**Theorem (A. Levine, 2014)**

There exist satellite operators that are injective but not surjective.
Fractals

What is left to show?

In order for \( C \) to be a fractal, we need some notion of distance or size, to see that we have smaller and smaller embeddings of \( C \) within itself.

One way to do this is to exhibit a metric space structure on \( C \). There are several natural metrics on \( C \), but we have not yet found one that works well with the current results on satellite operators. The search is on!
The origins of mathematical knot theory

1880s: Kelvin (1824–1907) hypothesized that atoms were ‘knotted vortices’ in æther. This led Tait (1831–1901) to start tabulating knots.

Tait thought he was making a periodic table!
Examples of knots

Figure: Knots in circular DNA.

(Images from Cozzarelli, Sumners, Cozzarelli, respectively.)