There exist infinitely many unknotted winding number one satellite operators on knot concordance

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Preliminaries

Definition

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A 4–dimensional equivalence relation on knots

Two knots $K$ and $J$ are said to be concordant if they cobound a properly embedded smooth annulus in $S^3 \times [0, 1]$. 

$S^3 \times [0, 1]$
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Two knots $K$ and $J$ are said to be **concordant** if they cobound a properly embedded smooth annulus in $S^3 \times [0, 1]$. 
The knot concordance group

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The identity element in $C$ is the class of the unknot. That is, the class of knots which bound smoothly embedded disks in $B^4$, called slice knots.
Variants of the knot concordance group

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Two knots are **topologically concordant** if they cobound a **topologically embedded** annulus in a manifold **homeomorphic** to $S^3 \times [0,1]$. Topological concordance classes of knots form the **topological knot concordance group**, denoted $C^{\text{top}}$. 
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Two knots are **exotically concordant** if they cobound a **smoothly embedded** annulus in a smooth manifold **homeomorphic** to $S^3 \times [0,1]$. Exotic concordance classes of knots form the **topological knot concordance group**, denoted $C^{\text{ex}}$. 

If the 4–dimensional (smooth) Poincaré Conjecture is true, $C = C^{\text{ex}}$. 
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Definition

The **winding number** of a pattern is the signed count of its intersections with a meridional disk of the solid torus.
The satellite construction

$P$, the pattern

$K$, a knot in $S^3$

**Figure**: The satellite operation on knots in $S^3$. 
The satellite construction

\( P \), the pattern \( K \), a knot in \( S^3 \) \( P(K) \), the satellite knot

**Figure**: The satellite operation on knots in \( S^3 \).
The satellite construction

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$P(K)$, the satellite knot

Figure: The satellite operation on knots in $S^3$.

Remark

Any satellite operator $P$ gives a function $P : C \to C$. 
Strong winding number one operators

Consider $P$ in $S^3$ instead of the solid torus. Call this $\tilde{P}$.

Definition

If $\eta$, the meridian of the solid torus, normally generates $\pi_1(S^3 \setminus \tilde{P})$, then $P$ is said to have strong winding number one.

For a $P$ such that $\tilde{P}$ is unknotted, $P$ is strong winding number one if and only if it is winding number one.
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Injectivity of satellite operators

Theorem (Cochran–Davis–R., ’12)

If $P$ is a strong winding number one pattern, then

$$P : C^{\text{top}} \to C^{\text{top}} \text{ and } P : C^{\text{ex}} \to C^{\text{ex}}$$

are injective. That is, for any two knots $K$ and $J$,

$$P(K) = P(J) \iff K = J$$
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If the 4–dimensional Poincaré Conjecture is true, $P : C \to C$ is injective.
Is $C$ a fractal?

A fractal can be defined as a set which ‘exhibits self-similarity on many scales’.
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Each strong winding number one satellite operator gives a ‘self-similarity’ of $C^{\text{top}}$ and $C^{\text{ex}}$ (and maybe even of $C$).
Is $C$ a fractal?

A fractal can be defined as a set which ‘exhibits self-similarity on many scales’.
Each strong winding number one satellite operator gives a ‘self-similarity’ of $C^{\text{top}}$ and $C^{\text{ex}}$ (and maybe even of $C$).

**Question**

How many strong winding number one operators are there?
Theorem (R.)

There is a strong winding number one satellite operator $P$ and a large family of knots $K$ such that $P^i(K) = P(P(\cdots(P(K))\cdots))$ are all distinct in $C^{ex}$ and $C$. That is, $P^i(K) \neq P^j(K)$ for all $i \neq j$.

Therefore, each $P^i$ gives a distinct function on the smooth knot concordance group.
Main theorem

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Therefore, each $P^i$ gives a distinct function on the smooth knot concordance group.
Each $P^i$ is strong winding number one. So we have infinitely many self-similarities of $C^{ex}$.
We can choose $K$ to be topologically slice and $\widetilde{P}$ to be unknotted, in which case the set $\{P^i(K)\}$ is an infinite family of topologically slice knots that are distinct in smooth concordance.
Ozsváth–Szabó defined the $\tau$–invariant of a knot. This gives homomorphisms $\tau : C \to \mathbb{Z}$ and $\tau : C^{\text{ex}} \to \mathbb{Z}$. 
The satellite construction

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\(\tau\)-invariant of knots

**Definition**

Ozsváth–Szabó defined the \(\tau\)-invariant of a knot. This gives homomorphisms \(\tau: \mathcal{C} \to \mathbb{Z}\) and \(\tau: \mathcal{C}^{\text{ex}} \to \mathbb{Z}\).

**Proposition (Ozsváth–Szabó)**

*Start with a knot \(K_+\). If \(K_-\) is the knot obtained by changing a single positive crossing of \(K_+\), then*

\[\tau(K_+) - 1 \leq \tau(K_-) \leq \tau(K_+)\]
Composition of patterns

\[ P(Q(K)) = (P \ast Q)(K) \]

**Figure**: The monoid operation on patterns.
Legendrian front diagrams

Every knot has a Legendrian front diagram, i.e. a diagram with no vertical tangencies wherein all crossings are of the following type:

\[ \times \]
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Classical invariants of Legendrian knots

$$\text{tb}(K) = (\# \text{positive crossings} - \# \text{negative crossings}) - \frac{1}{2} \# \text{cusps}$$

$$\text{rot}(K) = \frac{1}{2} (\# \text{down cusps} - \# \text{up cusps})$$
Classical invariants of Legendrian knots

\[ \text{tb}(K) = (\# \text{positive crossings} - \# \text{negative crossings}) - \frac{1}{2} \# \text{cusps} \]
\[ \text{rot}(K) = \frac{1}{2}(\# \text{down cusps} - \# \text{up cusps}) \]

\[ \text{tb}(K) = (3 - 0) - \frac{1}{2}(4) = 1, \quad \text{rot}(K) = \frac{1}{2}(2 - 2) = 0 \]
Classical invariants for Legendrian patterns

\[ \text{tb}(P) = 2 \quad \text{and} \quad \text{rot}(P) = 0 \]
For a knot $K$, suppose we have a Legendrian diagram with $tb(K) = 0$. 
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The Legendrian satellite operation
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For a knot $K$, suppose we have a Legendrian diagram with $tb(K) = 0$. We can obtain the satellite knot $P(K)$ by taking parallels of $K$ and then inserting the pattern.
Legendrian patterns and Legendrian satellites

**Proposition (Ng)**

\[ \text{tb}(P(K)) = \text{tb}(P) + w(P)^2 \text{tb}(K) \]

\[ \text{rot}(P(K)) = \text{rot}(P) + w(P) \text{rot}(K) \]
Legendrian patterns and Legendrian satellites

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Proposition

\[ \text{tb}(P \ast Q) = \text{tb}(P) + w(P)^2 \text{tb}(Q) \]

\[ \text{rot}(P \ast Q) = \text{rot}(P) + w(P)\text{rot}(Q) \]
The slice–Bennequin inequality

Slice–Bennequin inequality (Rudolph)

For any knot $K$, we have that

$$tb(K) + |rot(K)| \leq 2\tau(K) - 1$$
The proposition states that for any knot $K$ with $tb(K) = 0$, $rot(K) = 2\tau(K) - 1$ and $\tau(K) > 0$, $P(K) \neq K$ in $C$ (and therefore, in $C^{ex}$).

Note: There are large families of such knots $K$. 

**Proposition (Cochran–Franklin–Hedden–Horn)**

For any knot $K$ with $tb(K) = 0$, $rot(K) = 2\tau(K) - 1$ and $\tau(K) > 0$, $P(K) \neq K$ in $C$ (and therefore, in $C^{ex}$).
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Note: There are large families of such knots $K$.

**Proof:**

$tb(P(K)) = tb(P) + tb(K) = 0$ and 
$rot(P(K)) = rot(P) + rot(K) = 2 + (2\tau(K) - 1) = 2\tau(K) + 1$.

$tb(P) = 0$ and $rot(P) = 2$
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Proof: $tb(P(K)) = tb(P) + tb(K) = 0$ and $rot(P(K)) = rot(P) + rot(K) = 2 + (2\tau(K) - 1) = 2\tau(K) + 1$.

But $tb(P(K)) + |rot(P(K))| \leq 2\tau(P(K)) - 1$. 
Proof

\[ \text{tb}(P) = 0 \text{ and } \text{rot}(P) = 2 \]

Proposition (Cochran–Franklin–Hedden–Horn)

For any knot \( K \) with \( \text{tb}(K) = 0 \), \( \text{rot}(K) = 2\tau(K) - 1 \) and \( \tau(K) > 0 \), \( P(K) \neq K \) in \( C \) (and therefore, in \( C^{ex} \)).

Note: There are large families of such knots \( K \).

Proof: \( \text{tb}(P(K)) = \text{tb}(P) + \text{tb}(K) = 0 \) and \( \text{rot}(P(K)) = \text{rot}(P) + \text{rot}(K) = 2 + (2\tau(K) - 1) = 2\tau(K) + 1 \)
But \( \text{tb}(P(K)) + |\text{rot}(P(K))| \leq 2\tau(P(K)) - 1 \)
So \( 0 + 2\tau(K) + 1 \leq 2\tau(P(K)) - 1 \)
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*For any knot $K$ with $tb(K) = 0$, $rot(K) = 2\tau(K) - 1$ and $\tau(K) > 0$, $P(K) \neq K$ in $C$ (and therefore, in $C^{ex}$).*

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tb(P(K)) = tb(P) + tb(K) = 0 \quad \text{and} \quad rot(P(K)) = rot(P) + rot(K) = 2 + (2\tau(K) - 1) = 2\tau(K) + 1
\]

But $tb(P(K)) + |rot(P(K))| \leq 2\tau(P(K)) - 1$

So $0 + 2\tau(K) + 1 \leq 2\tau(P(K)) - 1 \Rightarrow \tau(K) + 1 \leq \tau(P(K))$
The satellite construction

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Proof

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Proposition (Cochran–Franklin–Hedden–Horn)

For any knot \( K \) with \( \text{tb}(K) = 0 \), \( \text{rot}(K) = 2\tau(K) - 1 \) and \( \tau(K) > 0 \), \( P(K) \neq K \) in \( \mathcal{C} \) (and therefore, in \( \mathcal{C}^{\text{ex}} \)).

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**Proof:**

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\text{tb}(P(K)) = \text{tb}(P) + \text{tb}(K) = 0 \quad \text{and} \\
\text{rot}(P(K)) = \text{rot}(P) + \text{rot}(K) = 2 + (2\tau(K) - 1) = 2\tau(K) + 1
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But \( \text{tb}(P(K)) + |\text{rot}(P(K))| \leq 2\tau(P(K)) - 1 \)

So \( 0 + 2\tau(K) + 1 \leq 2\tau(P(K)) - 1 \Rightarrow \tau(K) + 1 \leq \tau(P(K)) \)

\( \Rightarrow P(K) \neq K \)
Proof

Proposition (R.)

\[ P^i(K) \neq K \text{ for any } i > 0 \text{ in } C \text{ (and therefore, in } C^{ex}). \]
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[Diagram of connected lines]
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Figure: The operator \( P^2 \)
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Proof:

\[
\begin{align*}
\text{tb}(P^2) &= \text{tb}(P) + \text{tb}(P) \\
\text{rot}(P^2) &= \text{rot}(P) + \text{rot}(P)
\end{align*}
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Figure: The operator \( P^2 \)
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\[ \text{tb}(P^i) = 0 \text{ and } \text{rot}(P^i) = 2i \]
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\[ \text{tb}(P^i(K)) = 0 \quad \text{and} \quad \text{rot}(P^i(K)) = 2\tau(K) - 1 + 2i \]
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By the slice–Bennequin inequality, we have that
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By the slice–Bennequin inequality, we have that

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By the slice–Bennequin inequality, we have that

\[ \text{tb}(P^i(K)) + |\text{rot}(P^i(K))| \leq 2\tau(P^i(K)) - 1 \]
\[ 0 + |2\tau(K) - 1 + 2i| \leq 2\tau(P^i(K)) - 1 \]
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\text{tb}(P^i) &= 0 \text{ and } \text{rot}(P^i) = 2i \\
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By the slice–Bennequin inequality, we have that

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\text{tb}(P^i(K)) + |\text{rot}(P^i(K))| \leq 2\tau(P^i(K)) - 1 \\
0 + |2\tau(K) - 1 + 2i| \leq 2\tau(P^i(K)) - 1
\]

Therefore, \(\tau(K) + i \leq \tau(P^i(K))\) and \(P^i(K) \neq K\) for \(i > 0\). \(\square\)
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Proof

Theorem (R.)

\( P^i(K) \neq P^j(K) \) for any \( i \neq j \) in \( C \) (and therefore, in \( C^{ex} \)).

Additionally, \( \tau(P^i(K)) = \tau(K) + i \) for all \( i \geq 0 \).
Proof (R.)

$P^i(K) \neq P^j(K)$ for any $i \neq j$ in $\mathcal{C}$ (and therefore, in $\mathcal{C}^{ex}$).
Additionally, $\tau(P^i(K)) = \tau(K) + i$ for all $i \geq 0$

**Proof:** We can change $P^i(K)$ to $P^{i-1}(K)$ by changing a single positive crossing to a negative crossing. Therefore, we know that

$$\tau(P^{i-1}(K)) \leq \tau(P^i(K)) \leq \tau(P^{i-1}(K)) + 1$$
Proof: We can change $P^i(K)$ to $P^{i-1}(K)$ by changing a single positive crossing to a negative crossing. Therefore, we know that

$$\tau(P^{i-1}(K)) \leq \tau(P^i(K)) \leq \tau(P^{i-1}(K)) + 1$$

Therefore,

$$\tau(P^i(K)) \leq \tau(P^{i-1}(K)) + 1 \leq \tau(P^{i-2}(K)) + 2 \leq \cdots \leq \tau(K) + i.$$ 

$$\Rightarrow \tau(P^i(K)) = \tau(K) + i \text{ for all } i > 0$$