

Maps: creating surfaces from polygons

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Roughly speaking, a *map* is a surface obtained by gluing a set of polygons along their edges, as is done in Figure 1. Imagine, for instance, that you are given 100 squares of paper and that you start gluing pairs of square sides until you form a surface without boundary. You have just created one of the billions of possible maps with 100 squares. We could decide to choose one of these billions of maps at random and look at what the result look like: is this “random surface” roughly spherical? how far apart are the squares?, are there bottleneck? what topology does it have? These questions are all the more important that it is believed that the probability on random surfaces obtained by taking large random maps are relevant for the description of important physical phenomenon, such as the evolution of strings in string theorist. We could also wonder how to efficiently encode our map (the list of square-side’s gluings) in a computer. This is of practical importance because maps are used to represent objects in computer images (see Figure 3). There are also some connections between maps and permutations and random matrices that stems for the ability to encode many mathematical structures in terms of maps.

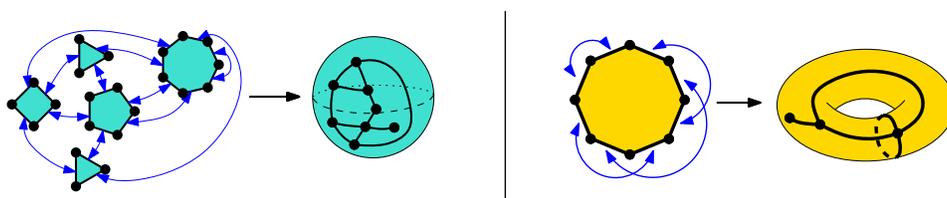


Figure 1: Two maps (seen as obtained by identifying edges of a set of polygons in pairs). The map on the left is a planar map with 5 faces. The map on the right is a map of genus 1 with 1 face.

We now give a more precise definition of maps and their relation to various question in mathematics and physics. A *map* is a decomposition of

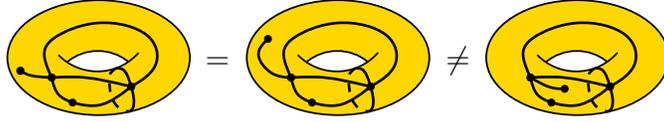


Figure 2: Maps seen as an embedding of a graph in a surface, considered up to homeomorphism. The two maps on the right are equal, but they are different from the map on the left.

a 2-dimensional connected surface into a finite number of *vertices*, *edges*, and *faces* (2-dimensional regions homeomorphic to a disc). Maps are considered up to homeomorphism. In Figure 1, we have represented two maps: a *planar map* (that is, a map on the sphere), and a *map of genus 1* (that is, a map on the torus). There are two ways to think about maps:

- The first way is to think about maps as surfaces obtained by identifying edges of a set of polygons in pairs. This way of thinking about map is represented in Figure 1.
- The second way is to think about maps as an embedding of a graph in a surface, considered up to homeomorphism. In this case, we require that the faces (the connected components of the complementary of the graph) are simply connected. This way of thinking about map is represented in Figure 2. Often, when representing planar maps, only the graph is represented (and not the underlying sphere) as is done in Figure 5.

The *degree* of a face is the number of incident edges. Familiar classes of maps are *triangulations* and *quadrangulations*, in which faces have degree 3 and 4 respectively. For instance, the map in Figure 3 is a planar triangulation.

The enumeration of maps has a long history, starting with the work of William Tutte in the sixties motivated by the 4-colors theorem. Tutte enumerated several classes of maps (triangulations, simple maps, etc) using some convoluted generating function techniques [52, 51, 53, 54] (see [10] for a modern account). The surprise, was that many classes of maps turned out to have elegant counting formulas reminiscent of formulas for trees. For instance, the number of rooted planar maps with n edges is $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$.

Maps were then considered (under different names such as *ribbon graphs*, *fat-graphs*, *cyclic graphs*, *graphs with a rotation system*, *combinatorial maps*, *Feynman diagrams*, *random lattices*) in many different fields of mathematics and theoretical physics. For instance,

- Theoretical physicists realized in the seventies that certain matrix integrals (for random matrices taken under the GUE law) could be in-

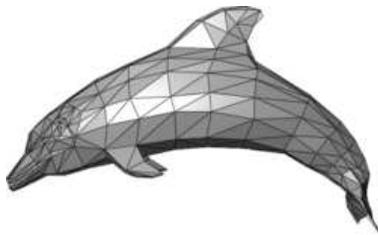


Figure 3: Meshes of surfaces: computer representation of surfaces as maps.

terpreted as generating functions of maps [8, 20, 50]; see [55] for an introduction. This correspondence is robust enough to write partition functions of many statistical mechanics models on maps as matrix integrals, and to use this interpretation to solve the models; see e.g. [26, 30].

- Combinatorialists used maps to represent products of permutations [31], or coverings of the sphere [32]. This representation was used to solve problems such as computing connection coefficients in the symmetric group [49, 7, 29, 11].
- Probabilists started using maps as a way of defining random surfaces. The idea is to consider uniformly random maps with n edges as a random metric space, and then take a scaling limit (in the Gromov-Hausdorff sense) in order to define a random surface known as the *Brownian map* [35, 37, 39, 42, 33]. This is akin to defining the Brownian motion as the limit of random lattice paths.
- Maps have practical applications in computer science: for representing surfaces on a computer screen (see Figure 3), and for designing graph drawing algorithms (see Figure 4). Maps also appear in connection with problems in graph theory (see [44]) and knot theory (see [32]).

At the turn of the century, a bijective approach to planar maps emerged with the seminal work of Schaeffer [48] (after earlier construction by Cori and Vauquelin [25] and Arquès [1]). Many bijections were subsequently obtained between classes of planar maps and classes of trees [5, 3, 14, 13, 19, 12, 11, 15, 16, 41, 47, 45, 46, 27] or classes of lattice paths [2, 4, 28]. This represented a breakthrough in the understanding of maps (and their simple-looking enumerative formulas), and allowed to tackle many new problems. Indeed, trees are easy to handle in order to count, encode, sample, or take scaling limits.

The bijective approach allowed to solve statistical model on maps [12, 15, 16]. It was also used to define efficient encoding algorithms for maps [22,

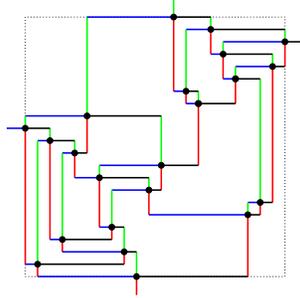


Figure 4: Orthogonal drawing of 4-regular plane graphs with one bend per edge obtained by using the algorithm described in [6].

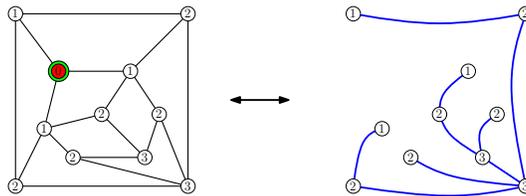


Figure 5: Schaeffer bijection between planar quadrangulations with a marked vertex and well-labeled trees. The numbers indicated on the vertices of the quadrangulation are the graph distance to the marked vertex.

21, 46, 27, 9]. The bijective approach to maps was also crucial to the study of the metric properties of random maps (here the term *metric* refers to the graph distance between vertices of the map) and their scaling limits. Indeed, one of the bijections of Schaeffer [48] establishes a correspondence between planar quadrangulations with a marked vertex and *well-labeled trees*, that is, plane trees where vertices have integer *labels* such that labels of adjacent vertices differ by at most one, and the minimum label is 1. Schaeffer bijection is represented in Figure 5. It has the following remarkable property: the non-marked vertices of the quadrangulation corresponds to the vertices of the tree, and the graph distance between a non-marked vertex v and the marked vertex in the quadrangulation corresponds to the label of the vertex v in the tree. This property allows one to study the metric properties of quadrangulations via the label properties of trees. This bijection (and its generalizations [14, 41, 23]) is at the root of the recent breakthrough in the study of maps as random metric spaces [24, 18, 19, 17, 43, 38, 35, 37, 36, 34, 39, 40, 42, 33].

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