

On the distance-profile of random rooted plane graphs

Olivier Bernardi^{1†} and Gwendal Collet^{2‡} and Éric Fusy^{2§}

¹Department of Mathematics, Brandeis University, Waltham, MA 02453, USA. bernardi@brandeis.edu

²LIX, École Polytechnique, 91128 Palaiseau, France. gcollet@lix.polytechnique.fr

Abstract. We study the distance-profile of the random rooted plane graph G_n with n edges (by a plane graph we mean a planar map with no loops nor multiple edges). Our main result is that the profile and radius of G_n (with respect to the root-vertex), rescaled by $(2n)^{1/4}$, converge to explicit distributions related to the Brownian snake. A crucial ingredient of our proof is a bijection we have recently introduced between rooted outer-triangular plane graphs and rooted eulerian triangulations, combined with ingredients from Chassaing and Schaeffer (2004), Bousquet-Mélou and Schaeffer (2000), and Addario-Berry and Albenque (2013). We also show that the result for plane graphs implies similar results for random rooted loopless maps and general maps.

Keywords: Planar maps, bijections, ISE, Brownian snake, distance-profile

1 Introduction

A *planar map* is a connected planar graph embedded in the plane considered up to deformation. Planar maps can be considered as metric spaces by defining the *distance* between vertices as the minimal number of edges of the paths joining them. The study of the metric properties of random planar maps has been a very active subject of research for the past 10 years. The first key results were obtained in the seminal paper Chassaing and Schaeffer (2004), focusing on the class of *rooted* quadrangulations (a map is rooted by marking a directed edge having the outer face on its right). It was shown there that typical distances in a uniformly random rooted quadrangulation Q_n with n vertices is of order $n^{1/4}$ (in contrast with the typical distance of $n^{1/2}$ for random plane trees), and that the distance-profile (the collection of distances of the n vertices from the root-vertex) converges in law to an explicit distribution related to the *ISE* (the *integrated superBrownian excursion* is the occupation measure of the Brownian snake introduced in Aldous (1993)). This result was then generalized to other classes of random maps, in particular to random rooted maps with n vertices and Boltzmann weights on the faces in Marckert and Miermont (2007); Miermont and Weill (2008).

[†]Supported by NSF grant DMS-1308441

[‡]Supported by ANR project Cartaplus

[§]Supported by ANR project Cartaplus

In this article, we prove this type of result on the profile for rooted simple maps, that is, maps without loops nor multiple edges, which are classically called *plane graphs*. We also show that it implies the same type of result for the class of loopless maps, and for the class of all maps. We now give a few definitions in view of stating our main result. For a rooted planar map $G = (V, E)$ with n edges, the *distance* $d(e)$ of an edge $e \in E$ (with respect to the root) is the length of a shortest path of G starting at e and ending at the root-vertex, the *distance-profile* of G is the n -set $\{d(e)\}_{e \in E}$ (note that this is a distance-profile at edges we consider, not at vertices). Let us now give some terminology for the type of convergence results to be obtained. We denote by \mathcal{M}_1 the set of probability measures on \mathbf{R} , endowed with the *weak topology* (that is, the topology given by the convergence in law). Note that ISE is a random variable taking its values in \mathcal{M}_1 . For $\mu \in \mathcal{M}_1$, denote by $F_\mu(x)$ the cumulative function of μ , $\inf(\mu) := \inf\{x : F_\mu(x) > 0\}$ and $\sup(\mu) := \sup\{x : F_\mu(x) < 1\}$, and define the *width* of μ as $\sup(\mu) - \inf(\mu)$. We also define the *nonnegative shift* of μ as the probability measure (with support in \mathbf{R}_+) whose cumulative function is $x \mapsto F_\mu(x + \inf(\mu))$.

Definition 1 A sequence $\mu^{(n)}$ of random variables taking values in \mathcal{M}_1 is said to satisfy the ISE limit property if the following properties hold, where μ_{ISE} is the random variable in \mathcal{M}_1 given by the ISE law:

- $\sup(\mu^{(n)})$ converges in law to the width of μ_{ISE} .
- $\mu^{(n)}$ converges in law to the nonnegative shift of μ_{ISE} (for the weak topology on \mathcal{M}_1).

For an n -set $\mathbf{x} = \{x_1, \dots, x_n\}$ of nonnegative values, and for $a > 0$, define $\mu_a(\mathbf{x})$ as the probability measure

$$\mu_a(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \delta_{x_i / (an)^{1/4}},$$

where δ_x denotes the Dirac measure at x . Our main result is the following:

Theorem 2 For $n \geq 1$, let π_n be the distance-profile of the uniformly random rooted plane graph with n edges. Then $\mu_2(\pi_n)$ satisfies the ISE limit property.

Relation with other work and perspectives. Although we focus here on the convergence of the profile of random maps, much stronger results have now been proved for several classes of maps. More precisely, for a given class \mathcal{C} of maps, one can consider the uniformly random map C_n of size n on maps as a random metric space. It is then natural to study the limit of the random metric space C_n (rescaled by $n^{1/4}$) in the Gromov Hausdorff topology. In a series of groundbreaking papers, this type of convergence was proved to hold for triangulations and $2p$ -angulations in Le Gall (2007, 2013) and independently for quadrangulations in Miermont (2013). The limit is a (continuous) random metric space called the *Brownian map*, which has almost surely spherical topology. Combining the techniques of Le Gall (2013) with some new ingredients, the same result was proved for simple triangulations and simple quadrangulations in Addario-Berry and Albenque (2013). At the moment we do not know if we can extend our proof for the profile of random simple maps in order to obtain convergence to the Brownian map.

We close this section by recalling a useful classical result. For μ and ν two elements of \mathcal{M}_1 , the *linear Wasserstein distance* between μ and ν is defined as

$$W_1(\mu, \nu) = \int_{\mathbf{R}} |F_\mu(x) - F_\nu(x)| dx,$$

which endows \mathcal{M}_1 with a metric structure. Another characterization of $W_1(\mu, \nu)$ is to be the infimum of $\mathbf{E}(|X - Y|)$ over all couplings (X, Y) where the law of X is μ and the law of Y is ν . It is known that if a sequence $\mu^{(n)}$ of elements of \mathcal{M}_1 converges to μ for the metric W_1 , then $\mu^{(n)}$ also converges to μ for the weak topology on \mathcal{M}_1 . Hence the following claim:

Claim 3 *Let $\mu^{(n)}$ and $\nu^{(n)}$ be two sequences of random variables in \mathcal{M}_1 (i.e., each variable is a random probability measure), living in the same probability space. Assume that $\mu^{(n)}$ satisfies the ISE limit property and that, for each fixed $\epsilon > 0$, $P(W_1(X_n, Y_n) \geq \epsilon)$ converges to 0 and $P(|\sup(\mu^{(n)}) - \sup(\nu^{(n)})| \geq \epsilon)$ converges to 0. Then $\nu^{(n)}$ satisfies the ISE limit property.*

2 Bijection between outer-triangular plane graphs and eulerian triangulations, and transfer of canonical paths

In this section we recall a bijection established in Bernardi et al. (2014) between outer-triangular plane graphs and eulerian triangulations, and establish a crucial property for canonical paths.

A rooted plane graph C is said to be *outer-triangular* if its outer face (that is, the root face, drawn as the infinite face in the planar representation of C) has degree 3. Given an outer-triangular plane graph G , a *3-orientation with buds* of G is an orientation of the inner edges of G (outer edges are left unoriented), with additional outgoing half-edges at inner vertices, called *buds*, such that each inner (resp. outer) vertex has outdegree 3 (resp. 0), and each inner face of degree $d+3$ has d incident buds⁽ⁱ⁾. It is shown in Bernardi and Fusy (2012) that each rooted outer-triangular plane graph G admits a unique 3-orientation with buds, called the *canonical 3-orientation*, satisfying the following properties:

- *Outer-accessibility*: there is a directed path from any inner vertex to a vertex of the outer face.
- *Minimality*: There is no clockwise circuit.
- *Local property at buds*: the first edge following each bud in clockwise order must be outgoing.

Figure 1.(a) shows such an outer-triangular plane graph endowed with its canonical 3-orientation.

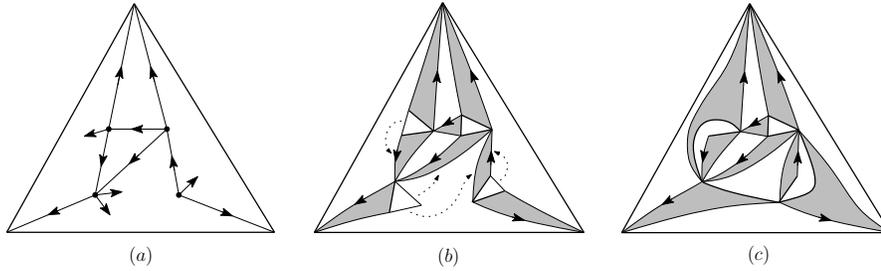


Fig. 1: (a) An outer-triangular plane graph endowed with its canonical 3-orientation with buds, (b) after inflation and (c) after merging, the resulting eulerian triangulation endowed with its canonical 1-orientation.

A *rooted eulerian triangulation* is a rooted planar map (which may have multiple edges) where each face has degree 3 and each vertex has even degree. Hence faces can be properly bicolored (in light or

⁽ⁱ⁾ When G is a maximal plane graph, 3-orientations have no bud, and correspond to the well-known Schnyder structures, introduced in Schnyder (1989).

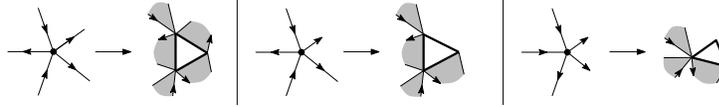
dark) such that each light (resp. dark) face is adjacent only to dark (resp. light) faces. By convention (since there are exactly two possible colorings), the root face is dark.

As shown in Bernardi et al. (2014), there exists a bijection between these two families of maps. We just recall here how to obtain a rooted eulerian triangulation from a rooted outer-triangular plane graph. Let C be an outer-triangular plane graph endowed with its canonical 3-orientation with buds. The bijection illustrated in Figure 1 has two steps: *inflation*, then *merging*. First, inner edges and inner vertices will be inflated in the following way:

- Each inner edge becomes a dark triangle as indicated below.



- Each inner vertex becomes a light triangle whose edges correspond to the outgoing half-edges (including buds). The cases with 0,1 or 2 buds are illustrated below.



After inflation, former inner faces of degree $d+3$ ($d \geq 0$) have now degree $2d+3$ (the d incident buds have turned into edges). Considering edges coming from buds as opening parenthesis, and remaining edges as closing parenthesis, one can form a clockwise parenthesis system leaving 3 edges unmatched. Hence, after merging the matched edges, the 3 unmatched edges form a light triangle. This ensures that each face of the resulting map is a triangle; see Figure 1. Moreover, the edges created by the inflation are incident to a dark and a light face, except for edges coming from buds, which are incident to two light faces. After merging, these edges are necessarily incident to a dark face as well. Therefore the triangulation is properly bicolored and is an eulerian triangulation (the outer face, which is left unchanged, is colored dark).

Let C be an outer-triangular rooted plane graph. For each inner edge e , we define its *canonical path* $P(e)$ to be the directed path in the canonical 3-orientation of C starting at e and following the rightmost (with respect to the previous edge on the path) outgoing edge until reaching a vertex of the outer face ($P(e)$ exist because the canonical 3-orientation is minimal and outer-accessible). Let G be a rooted eulerian triangulation. A *1-orientation* of G is an orientation of some inner edges (outer edges are left unoriented), such that each dark inner face has one directed edge, which is counterclockwise, and each inner (resp. outer) vertex has outdegree 1 (resp. 0). In a similar fashion as for outer-triangular plane graphs, G has a unique 1-orientation, called its *canonical 1-orientation*, which satisfies *outer-accessibility*; see Figure 1(c). For each inner vertex v , we define its *canonical path* $P(v)$ to be the unique directed path in the canonical 1-orientation starting at v and following the next outgoing edge until reaching a vertex of the outer face.

To an edge e of an outer-triangular rooted plane graph, we denote $\sigma(e)$ the inner vertex of the associated rooted eulerian triangulation which is the origin of e after inflation of the edge e into a dark triangle.

Proposition 4 *Let C be an outer-triangular rooted plane graph and G be the associated eulerian triangulation. The mapping σ gives a bijection between the edges of C and the inner vertices of G . Moreover the canonical path of an edge e of C has the same length as the canonical path of the vertex $\sigma(e)$ of G .*

Proof: The canonical 3-orientation of C gives a partial orientation Ω of G . It is not hard to see that Ω a 1-orientation, therefore σ gives a bijection between the edges of C and the inner vertices of G . Moreover all the canonical paths of C are directed paths in the orientation Ω (that is, they are preserved by the inflation process), thus Ω is outer-accessible. Thus Ω is the canonical 1-orientation of G , and the canonical path of e becomes the canonical path of $\sigma(e)$. This completes the proof. \square

Remark 5 Beside the bijection between outer-triangular plane graphs and eulerian triangulations, we will also use the following injection between rooted outer-triangular plane graphs and rooted simple triangulations. Given a rooted outer-triangular plane graph C endowed with its canonical 3-orientation, every inner face of degree $d + 3$ contains d buds. We consider the triangulation T obtained from C by triangulating each inner face by completing the buds into complete edges and gluing these edges in counter-clockwise order around each inner face; see Figure 2. The 3-orientation of C gives a 3-orientation of T (which implies that T is simple) and we contend that it is the canonical 3-orientation of T . This follows easily from the fact that the canonical paths of C are canonical paths in T (because of the local property at buds) and therefore no clockwise cycle exist.

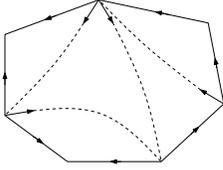


Fig. 2: Generic situation in a face of degree 7 with its 4 incident buds, and its canonical triangulation (dashed lines).

3 The profile of random rooted eulerian triangulations

3.1 Profile with respect to the root-vertex

Let T be a rooted eulerian triangulation with $n + 1$ vertices, let V be the vertex-set and let v_0 be the root-vertex. Recall that the faces of T are properly bicolored with the outer face being dark. A path P from a vertex v to a vertex v' is called *admissible* if each traversed edge of P has a dark face on its left. Let $\ell(v)$ be the length of a shortest admissible path from v to v_0 . The n -set $\{\ell(v)\}_{v \in V \setminus v_0}$ is called the *root-vertex profile* of T .

Proposition 6 Let π_n be the root-vertex profile of a uniformly random rooted eulerian triangulation with $n + 1$ vertices. Then $\mu_2(\pi_n)$ satisfies the ISE limit property.

Proof: It is shown in Bouttier et al. (2004) that random rooted eulerian triangulations with $n + 1$ vertices are in bijection with so-called *very well-labelled trees* with n nodes, i.e., rooted plane trees with n nodes, each node having a positive label such that adjacent node labels differ by 1 in absolute value, and the root is at a node of label 1. In addition the root-vertex profile of the eulerian triangulation corresponds to the n -set of labels of the corresponding tree. Hence π_n is distributed as the n -set of labels of the random very well-labelled tree with n nodes. The article Le Gall (2006) ensures ⁽ⁱⁱ⁾ that $\mu_2(\pi_n)$ satisfies the ISE limit property. \square

⁽ⁱⁱ⁾ See in particular Theorem 8.2 where the methodology is applied to the very close model where adjacent node labels differ by at most 1 in absolute value.

Remark 7 Alternatively one could prove Proposition 6 by recycling the combinatorial arguments from Sections 4.4 and 4.5 in Chassaing and Schaeffer (2004). This would require a detour via a model of “blossoming trees” (actually the one used in Bousquet-Mélou and Schaeffer (2000)) in order to drop the condition that the labels are positive.

3.2 Profile with respect to the outer face

Let T be a rooted eulerian triangulation, and let V be its set of inner vertices. For $v \in V$, we denote by $\tilde{d}(v)$ the length of canonical path of v . The set $\{\tilde{d}(v)\}_{v \in V}$ is called the *root-face profile* of T . Then it is easily checked from Bousquet-Mélou and Schaeffer (2000) that $\tilde{d}(v)$ is the length of a shortest admissible path from v to (a vertex of) the root-face. Hence $\ell(v) - 2 \leq \tilde{d}(v) \leq \ell(v)$. Thus Proposition 6 immediately gives (via Claim 3) the following result.

Proposition 8 Let π_n be the root-face profile of a uniformly random rooted eulerian triangulation with n inner vertices. Then $\mu_2(\pi_n)$ satisfies the ISE limit property.

4 The profile of random rooted plane graphs

4.1 Profile of random rooted outer-triangular plane graphs

Let G be a rooted outer-triangular plane graph, and let E_i be its set of inner edges. For $e \in E_i$ we denote by $\tilde{d}(e)$ the length of the canonical path of e , and by $d(e)$ the length of a shortest path starting at e and ending at (a vertex of) the root-face. The set $\{\tilde{d}(e)\}_{e \in E_i}$ is called the *canonical path profile* of G , and the set $\{d(e)\}_{e \in E_i}$ is called the *distance-profile at inner edges* of G . By Proposition 4 the canonical path profile of G coincides with the root-face profile of the rooted eulerian triangulation associated with G by the bijection of Section 2. Thus Proposition 8 gives:

Proposition 9 Let π_n be the canonical path profile of a uniformly random rooted outer-triangular plane graph with $n + 3$ edges. Then $\mu_2(\pi_n)$ satisfies the ISE limit property.

We will now prove that with high probability the canonical path profile is close from the distance-profile using the following (non-random) result from Addario-Berry and Albenque (2013).

Lemma 10 (Addario-Berry and Albenque (2013)) *There exists positive constants k_1, k_2 such that the following holds. Let G be a rooted simple triangulation, let e be an inner edge of G , let P be the canonical path of e (for the canonical 3-orientation of G), and let Q be another path from the origin of e to the root-face. If the length d of P is greater than the length d' of Q , then there exists a cycle C contained in $P \cup Q$ of length at most $k_1 d' / (d - d')$ such that each of the two parts of G resulting from cutting along C contains a (consecutive) subpath of Q of length at least $k_2(d - d')$.*

This implies the following (non-random) statement for rooted plane graphs, where the *diameter* $\text{Diam}(G)$ of a graph G is the maximal distance between pairs of vertices.

Lemma 11 *The statement of Lemma 10 also holds if one replaces “simple triangulation” by “outer triangular plane graph”. Consequently, for any $\Delta > 0$, if G is a rooted outer-triangular plane graph G , and e is an inner edge such that $d(e) \leq \tilde{d}(e) - \Delta$, then G has a cycle C of length at most $k_1 d(e) / \Delta$ such that the two parts G_ℓ, G_r resulting from cutting along C each have diameter at least $k_2 \Delta$.*

Proof: Let G be a rooted outer triangular plane graph. We consider its minimal 3-orientation with buds. As explained in Remark 5, there is a canonical way to complete the buds of G into complete edges so as to triangulate each inner face of G and obtain a simple triangulation \hat{G} endowed with its minimal 3-orientation. Moreover, for any inner edge e of G canonical path of e is the same in G and in \hat{G} . This proves the first statement. The second statement is a simple consequence obtained by considering the canonical path P of an edge e of G and a geodesic path Q . In this case G has a cycle C of length at most $k_1 d(e)/(\tilde{d}(e) - d(e)) \leq k_1 d(e)/\Delta$ such that the two parts G_ℓ, G_r resulting from cutting along C each have a subpath of Q of length at least $k_2(\tilde{d}(e) - d(e)) \geq k_2 \Delta$. Since a subpath of a geodesic path is geodesic, we conclude that each of G_ℓ, G_r has a diameter at least $k_2 \Delta$. \square

Definition 12 A sequence X_n of real random variables is said to have the uniform exponential decay property if there exist constants $a, b > 0$ such that for all n , $P(X_n \geq x) = a \exp(-bx)$.

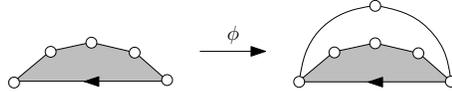
Lemma 13 Let H_n be the uniformly random rooted outer-triangular plane graph with $n + 3$ edges. Then $\text{Diam}(H_n)/n^{1/4}$ satisfies the uniform exponential decay property.

Proof: The property is inherited from eulerian triangulations. Precisely, let π_n denote the root-vertex profile of the uniformly random rooted eulerian. The calculations done in Section 6.2 of Chassaing and Schaeffer (2004) for well-labeled trees (which correspond to rooted quadrangulations) can be adapted verbatim to very well-labelled trees (which correspond to rooted eulerian triangulations) in order to show that $\sup(\mu_2(\pi_n))$ has the uniform exponential decay property. Hence, if π'_n denotes the root-face profile of the uniformly random rooted eulerian triangulation with n vertices, then $\sup(\mu_2(\pi'_n))$ has the uniform exponential decay property. This is in turn transferred (bijectively) to $\sup(\mu_2(\pi''_n))$, where π''_n is the canonical path profile of H_n . Then, since $d(e) \leq \tilde{d}(e)$, the property is also satisfied by $\sup(\mu_2(\pi'''_n))$, where $\pi'''_n = \{d(e)\}_{e \in E_i}$ is the distance-profile at inner edges of H_n . Since $\text{Diam}(H_n) \leq 2 \cdot \max_{e \in E_i} (d(e))$, we conclude that $\text{Diam}(H_n)/n^{1/4}$ satisfies the uniform exponential decay property. \square

For $n \geq 0$ we denote by \mathcal{M}_n the set of rooted plane graphs with n edges, and \mathcal{C}_n the subset of outer-triangular maps. It was shown in Bernardi et al. (2014) that

$$|\mathcal{C}_n| = 3 \cdot 2^{n-1} \frac{(2n)!}{n!(n+2)!} = O(8^n n^{-5/2}). \quad (1)$$

Observe that there is an injective map ϕ from \mathcal{M}_n to $\mathcal{M}_{n+2}^{(3)}$ as shown in the figure below.



Thus $|\mathcal{M}_n| \leq |\mathcal{C}_{n+2}| = O(8^n n^{-5/2})$. Moreover $\text{Diam}(G) = \text{Diam}(\phi(G))$. Thus (observing that $\frac{|\mathcal{M}_n|}{|\mathcal{C}_{n+2}|} \geq \frac{|\mathcal{C}_n|}{|\mathcal{C}_{n+2}|}$ is bounded away from 0) Lemma 13 implies the following.

Corollary 14 Let G_n be the uniformly random plane graphs with n edges. Then $\text{Diam}(G_n)/n^{1/4}$ satisfies the uniform exponential decay property. Therefore there exists constants $a, b > 0$ such that for all $n \geq 0$ and $x > 0$ the number of elements in \mathcal{M} of diameter at least $xn^{1/4}$ is at most $a8^n n^{-5/2} \exp(-bx)$.

Lemma 15 *Let $\epsilon > 0$ and let G_n be the random rooted outer-triangular plane graph with n edges. Let $\mathcal{E}_{n,\epsilon}$ be the event that G_n has an edge e for which $d(e) \leq \tilde{d}(e) - \epsilon n^{1/4}$. Then $\lim_{n \rightarrow \infty} P(\mathcal{E}_{n,\epsilon}) = 0$.*

Proof: We first show the statement for the event $\mathcal{E}_{n,\epsilon,A} = \mathcal{E}_{n,\epsilon} \cap \{\text{Diam}(G_n) \leq An^{1/4}\}$, where A is an arbitrary fixed positive constant. Let U_n be the set of rooted outer-triangular plane graphs with n edges of diameter at most $An^{1/4}$ having an edge e for which $d(e) \leq \tilde{d}(e) - \epsilon n^{1/4}$. By (1), it suffices to show that $|U_n| = o(8^n n^{-5/2})$. By Lemma 11 (applied to $\Delta = \epsilon n^{1/4}$), any map in U_n has a cycle of length $c \leq k_1 d(e)/\epsilon n^{1/4} \leq k_1 A/\epsilon$ separating two maps of diameter at least $k_2 \epsilon n^{1/4}$. We now fix a positive integer c , and denote V_n^c the set of pairs (G, C) where G is a rooted outer-triangular plane graph with n edges and C is a cycle of G of length c such that the two parts of G obtained by cutting along C each have diameter at least $k_2 \epsilon n^{1/4}$. It suffices to prove that $|V_n^c| = o(8^n n^{-5/2})$. Let $w_{i,n}$ be the number of maps in \mathcal{M}_i of diameter at least $k_2 \epsilon n^{1/4}$. By Corollary 14 there are constants $a, b' > 0$ such that $w_{i,n} \leq a 8^i i^{-5/2} \exp(-b'(n/i)^{1/4})$. Decomposing pairs $(G, C) \in V_n^c$ into two maps gives

$$|V_n^c| \leq \sum_{i+j=n+c} 2n \cdot w_{i,n} w_{j,n}$$

where the factor $2n$ accounts for choosing the position of the root edge of G . Let S be the above sum restricted to $\{i > n/(\log(n)^8)\} \cap \{j > n/(\log(n)^8)\}$ and S' the sum of the other terms. Since $w_{i,n} \leq a 8^i i^{-5/2}$,

$$S \leq (n+c) \cdot 2n \cdot a^2 8^{n+c} (n/\log(n)^8)^{-5} = o(8^n n^{-5/2}).$$

And since $w_{i,n} \leq a 8^i \exp(-b'(n/i)^{1/4})$,

$$S' \leq 2n/(\log(n)^8) \cdot 2n \cdot a^2 8^{n+c} \exp(-b'(\log(n))^2) = o(8^n n^{-5/2}).$$

Hence $|V_n^c| = o(8^n n^{-5/2})$ and this completes the proof that for any $A > 0$, $\lim_{n \rightarrow \infty} P(\mathcal{E}_{n,\epsilon,A}) = 0$. Thus for all $A > 0$,

$$\lim_{n \rightarrow \infty} P(\mathcal{E}_{n,\epsilon}) \leq \lim_{n \rightarrow \infty} (P(\mathcal{E}_{n,\epsilon,A}) + P(\text{Diam}(G_n) > An^{1/4})) \leq \sup_n P(\text{Diam}(G_n) > An^{1/4}).$$

And since by Lemma 13, $\lim_{A \rightarrow \infty} \sup_n P(\text{Diam}(G_n) > An^{1/4}) = 0$, we get $\lim_{n \rightarrow \infty} P(\mathcal{E}_{n,\epsilon}) = 0$. \square

Remark 16 *A result similar to Lemma 15 is given in Addario-Berry and Albenque (2013) for random rooted simple triangulations. However, we could not deduce Lemma 15 from that result and instead had to start from Lemma 10 above.*

We can now prove the main result of this section.

Proposition 17 *Let π_n be the distance-profile at inner edges of a uniformly random rooted outer-triangular plane graph with $n + 3$ edges. Then $\mu_2(\pi_n)$ satisfies the ISE limit property.*

Proof: Let G_n be the uniform random rooted outer-triangular plane graph with n inner edges, and let E_i be the set of inner edges. We consider the n -sets $\mathbf{d} = \{d_e\}_{e \in E_i}$ and $\tilde{\mathbf{d}} = \{\tilde{d}(e)\}_{e \in E_i}$. When $\mathcal{E}_{n,\epsilon}$ does not hold, then $W_1(\mu_2(\mathbf{d}), \mu_2(\tilde{\mathbf{d}})) \leq \epsilon/2^{1/4}$, and $|\sup(\mu_2(\mathbf{d})) - \sup(\mu_2(\tilde{\mathbf{d}}))| \leq \epsilon/2^{1/4}$. Hence, the result follows from Proposition 9 and Lemma 15, using Claim 3. \square

4.2 Profile of random rooted plane graphs

We now transfer our result for outer-triangular plane graphs to general plane graphs. For this we exploit an easy decomposition (already described in Bernardi et al. (2014)) of rooted plane graphs in terms of rooted outer-triangular plane graphs. Let \mathcal{M} be the family of rooted plane graphs, and let \mathcal{C} be the family of rooted outer-triangular plane graphs. Let p be the rooted plane graph with two edges meeting at a point, which is the root-vertex, and let $\mathcal{D} = \mathcal{C} \cup \{p\}$. For an element of \mathcal{D} , the *right-edge* is the edge following the root-edge in counterclockwise order around the root-face. It is shown in Bernardi et al. (2014) that each graph $\gamma \in \mathcal{M}$ is uniquely obtained from a sequence $\gamma_1, \dots, \gamma_k$ of elements of \mathcal{D} where the following operations are performed:

- (i) for $i \in [1..k-1]$, merge the right-edge of γ_i with the root-edge of γ_{i+1} (identifying the root-vertices),
- (ii) delete the right-edge of γ_k .

In the decomposition, γ_i (if it exists, i.e., if $i \leq k$) is called the i th component. This decomposition also ensures that the generating functions $M(z)$ of \mathcal{M} and $C(z)$ of \mathcal{C} (according to the number of edges) are related by

$$M(z) = \sum_{k \geq 1} (z + C(z)/z)^k = \frac{D(z)}{1 - D(z)}, \quad \text{where } D(z) := z + C(z)/z.$$

Let G_n be the random rooted plane graph with n edges, and for $i, j \geq 1$, let $\mathcal{E}_n^{(i,j)}$ be the event that, in the decomposition $\gamma_1, \dots, \gamma_k$ of G_n , the i th component γ_i exists (i.e., $i \leq k$) and has $n - j + 1$ edges. And let $\pi_n^{(i,j)}$ be the probability that $\mathcal{E}_n^{(i,j)}$ occurs.

Lemma 18 *For any $i, j \geq 0$, there exists a non-negative constant $\pi^{(i,j)}$ such that $\pi_n^{(i,j)}$ converges to $\pi^{(i,j)}$. In addition $\sum_{i,j} \pi^{(i,j)} = 1$.*

Proof: Let m_n be the number of rooted plane graphs with n edges, $m_n^{(i,j)}$ the number of rooted plane graphs with n edges for which $\mathcal{E}_n^{(i,j)}$ occurs (note that $\pi_n^{(i,j)} = m_n^{(i,j)}/m_n$), and d_n be the number of elements of \mathcal{D} with n edges. From the explicit expression

$$C(z) = \sum_{n \geq 1} \frac{3 \cdot 2^{n-1} (2n)!}{n!(n+2)!} z^{n+2} = \frac{z^2(-1 + 12z + \sqrt{1-8z})}{(1 + \sqrt{1-8z})^2}$$

it is easy to find that $D(z)$ and $M(z)$ have the following singular expansion at $z = 1/8$ (with the notation $Z = \sqrt{1-8z}$):

$$D(z) = \frac{5}{32} - \frac{9}{32}Z^2 + \frac{1}{4}Z^3 + O(Z^4), \quad M(z) = \frac{5}{27} - \frac{32}{81}Z^2 + \frac{256}{729}Z^3 + O(Z^4),$$

which we rewrite, with $d := 5/32$ and $e = 1/4$, as

$$D(z) = d + eZ^3 - 9Z^2/32 + O(Z^4), \quad M(z) = \frac{d}{1-d} + \frac{e}{(1-d)^2}Z^3 - 32Z^2/81 + O(Z^4).$$

Now, let $M^{(i,j)}(z) = \sum_n m_n^{(i,j)} z^n$. It is easy to see that $M^{(i,j)}(z) = a^{(i,j)} z^j D(z)$, where $a^{(i,j)} = [z^j] \frac{D(z)^{i-1}}{1-D(z)}$ accounts for the choice of the components γ_s for $s \neq i$. Hence $M^{(i,j)}(z)$ has a singular

expansion of the form $M^{(i,j)}(z) = d^{(i,j)} + e^{(i,j)}Z^{3/2} + g^{(i,j)}Z^2 + O(Z^4)$, with $e^{(i,j)} = a^{(i,j)} \cdot e \cdot 8^{-j}$. By classical transfer lemmas of singularity analysis in Flajolet and Sedgewick (2009),

$$m_n \sim \frac{1}{\sqrt{\pi}} \frac{e}{(1-d)^2} 8^n n^{-5/2}, \quad m_n^{(i,j)} \sim \frac{1}{\sqrt{\pi}} a^{(i,j)} \cdot e \cdot 8^{-j} \cdot 8^n n^{-5/2}.$$

Hence $\pi_n^{(i,j)} = m_n^{(i,j)} / m_n$ converges to the constant $\pi^{(i,j)}$ given by

$$\pi^{(i,j)} := (1-d)^2 8^{-j} [z^j] \frac{D(z)^{i-1}}{1-D(z)}.$$

We have for each $i \geq 1$, $\sum_j \pi^{(i,j)} = (1-d)^2 \cdot F(1/8)$, where $F(z) = D(z)^{i-1} / (1-D(z))$. Since $F(1/8) = D(1/8)^{i-1} / (1-D(1/8)) = d^{i-1} / (1-d)$, we conclude that $\sum_{i,j} \pi^{(i,j)} = 1$. \square

Lemma 19 *For $i, j \geq 0$ fixed, let π_n be the profile of the random rooted plane graph $G_n^{(i,j)}$ with n edges conditioned on $\mathcal{E}_n^{(i,j)}$. Then $\mu_2(\pi_n)$ satisfies the ISE limit property.*

Proof: Let E be the set of edges of $G_n^{(i,j)}$ and let E_i be the set of inner edges of γ_i . Let $\mathbf{d} = \{d_e\}_{e \in E}$ be the n -set of distances of the edges of $G_n^{(i,j)}$ from the root-vertex, and let $\mathbf{d}' = \{d_e\}_{e \in E_i}$ be the $(n-j-2)$ -set of distances of inner edges of γ_i from the root-vertex of γ_i (which is also the root-vertex of $G_n^{(i,j)}$). It is easy to see that there exists a constant $A > 0$ (depending only on i and j) such that, for any rooted plane graph with n edges and satisfying $\mathcal{E}_n^{(i,j)}$,

$$W_1(\mu_2(\mathbf{d}), \mu_2(\mathbf{d}')) \leq A \cdot \frac{\text{Diam}(\gamma_i)}{n}.$$

Since γ_i is a uniformly random rooted outer-triangular plane graph with $n-j+1$ edges, Lemma 13 ensures that $\text{Diam}(\gamma_i)/n^{1/4}$ satisfies the uniform exponential decay property hence

$$P(W_1(\mu_2(\mathbf{d}), \mu_2(\mathbf{d}')) \geq A/\sqrt{n}) = O(\exp(-\Omega(n^{1/4}))).$$

Similarly

$$P(|\text{sup}(\mu_2(\mathbf{d})) - \text{sup}(\mu_2(\mathbf{d}'))| \geq A/\sqrt{n}) = O(\exp(-\Omega(n^{1/4}))).$$

Since $\mu_2(\mathbf{d}')$ satisfies the ISE limit property according to Proposition 17, we conclude from Claim 3 that $\mu_2(\mathbf{d})$ also satisfies the ISE limit property. \square

Proof of Theorem 2. Let $\eta > 0$. Let k be the smallest value such that $\sum_{i \leq k, j \leq k} \pi^{(i,j)} > 1 - \eta$, and let $\mathcal{E}_{n,\eta}$ be the event that $\mathcal{E}_n^{(i,j)}$ holds for some $i \leq k$ and $j \leq k$. By Lemma 19, conditioned on $\mathcal{E}_{n,\eta}$, the random rooted plane graph with n edges satisfies the ISE limit property. Note that, as $n \rightarrow \infty$ the probability that $\mathcal{E}_{n,\eta}$ holds converges to $c_\eta := \sum_{i \leq k, j \leq k} \pi^{(i,j)}$ (because for n large enough two events $\mathcal{E}_n^{(i,j)}$ and $\mathcal{E}_n^{(i',j')}$ do not intersect), hence for n large enough, the probability that $\mathcal{E}_{n,\eta}$ holds is at least $1 - \eta$. Taking η arbitrarily small, we conclude that G_n satisfies the ISE limit property. \square

We define the *radius* $r(G)$ of a planar map G as the largest possible distance of a vertex of G from the root-vertex.

Proposition 20 *Let R_n be the radius of the random rooted plane graph G_n with n edges. Then $R_n/(2n)^{1/4}$ converges in law to the width of μ_{ISE} , and the convergence also holds for the moments.*

Proof: The convergence in law follows immediately from Theorem 2. The convergence of the moments then follows from the uniform exponential decay property of $R_n/(2n)^{1/4}$ which is given by Corollary 14. \square

5 The profile of two other map families

In this section we prove that Theorem 2 and Proposition 20 imply similar results for the class of rooted loopless maps, and the class of general rooted maps. The key tool (proved in Gao and Wormald (1999); Banderier et al. (2001)) is that a rooted loopless map has almost surely a “giant” simple component of linear size (concentrated around $2n/3$), and a rooted map has almost surely a “giant” loopless component of linear size (concentrated around $2n/3$). Some details are omitted by lack of space.

5.1 Profile of random rooted loopless maps

It is well known that a rooted loopless map M decomposes as a rooted simple map where each edge e is either left alone or one patches an arbitrary loopless map M_e at e (so that e and the root-edge of M_e have the same extremities). One can recursively apply the same procedure to each of the substituted loopless maps M_e , which in the end yields a *tree-decomposition* of M where the nodes correspond either to multiple edges of M or to rooted simple maps, and each edge of the tree corresponds to an edge of a simple component being part of a multiple edge of M .

We denote G_n the uniformly random rooted loopless map with n edges. Note that upon conditioning on the size of the simple maps appearing in the tree-decomposition of G_n , these simple maps are uniformly random (for their prescribed size) and independent.

Lemma 21 *There exists constants α, β such that for all $\epsilon > 0$, and all $n \geq 0$, $P(\text{Diam}(G_n) \geq n^{1/4+\epsilon}) \leq \alpha \exp(-n^{\beta\epsilon})$.*

Proof: By Corollary 14 there exist $a, b > 0$ such that the probability that each simple component of G_n has diameter greater than $n^{1/4+\epsilon}$ is at most $a \exp(-bn^\epsilon)$. Thus the probability that one of these components has diameter greater $n^{1/4+\epsilon}$ is at most $na \exp(-bn^\epsilon) < a' \exp(-n^{b'\epsilon})$ for some $a', b' > 0$. Now, let τ be the tree of the tree-decomposition of G_n . Using the arguments of Lemma 4.8 in Chapuy et al. (2010), one easily proves that there exist $a'', b'' > 0$ such that for all $\epsilon > 0$, $P(\text{Diam}(\tau) > n^\epsilon) < a'' \exp(-n^{b''\epsilon})$. Since $\text{Diam}(G_n) \leq 2 \cdot D \cdot \text{Diam}(\tau)$, we easily conclude. \square

Given a rooted loopless map G_n with n edges, we now define some events for G_n . We define \mathcal{E}_n as the event that the largest simple component B of G_n has its number of edges in the interval $[2n/3 - n^{3/4}, 2n/3 + n^{3/4}]$, and the loopless components attached at each of the edges of B are all of size (number of edges) at most $n^{3/4}$; from now on we call *loopless B-components* of G_n these components. Assuming that \mathcal{E}_n holds we define \mathcal{E}'_n as the event that the diameter of all loopless B -components is at most $n^{7/32}$, and the diameter of B is at most $n^{1/4} \log(n)^2$. Let E be the set of edges of G_n and E_B the set of edges of B . We define the *root-edge* of B as the edge of B bearing the loopless B -component containing the root-edge of G_n ; call *root-vertex of B* the origin of this edge endowed with an arbitrary orientation. For each edge $e \in E_B$ denote by $d_B(e)$ the length of a shortest path in B starting from e and ending at

the root-vertex of B . For each edge $e \in E$, let e_B be the edge of B bearing the loopless B -component containing e , define $d_B(e) := d_B(e_B)$. For each $i \geq 1$ denote by $E(i)$ the set of edges of E such that $d_B(e) \leq i$, and denote by $E_B(i)$ the set of edges of B such that $d_B(e) \leq i$. We define \mathcal{E}_n''' as the event $|E(i)| \in [\frac{|E_B|}{|E|}|E_B(i)| - n^{3/4}, \frac{|E_B|}{|E|}|E_B(i)|/3 + n^{3/4}] \forall i \geq 1$. Lastly, we define \mathcal{E}_n^* as the event that all of $\mathcal{E}_n, \mathcal{E}_n', \mathcal{E}_n''$ hold.

Lemma 22 *The probability of \mathcal{E}_n^* is $1 - O(\exp(-\Omega(\log(n)^2)))$.*

Proof: Using the asymptotic estimates in Gao and Wormald (1999); Banderier et al. (2001) (see also Lemma 3.7 in Chapuy et al. (2010)), it can be proved that $P(\mathcal{E}_n) > 1 - a \exp(-n^\delta)$ for some $a, \delta > 0$. Now if \mathcal{E}_n occurs, Corollary 14 and Lemma 21 easily prove that $P(\mathcal{E}_n') > 1 - a' \exp(-n^{\delta'})$ for some $a', \delta' > 0$. Lastly if \mathcal{E}_n occurs,

$$d_B(e) \leq d(e) \leq d_B(e) + 2n^{7/32} \quad \text{for all } e \in E.$$

Moreover, asymptotic estimates in Gao and Wormald (1999); Banderier et al. (2001) that, conditioned on \mathcal{E}_n ,

$$P(|E(i)| \in [2|E_B(i)|/3 - n^{1/2+\delta}, 2|E_B(i)|/3 + n^{1/2+\delta}]) = 1 - O(\exp(-n^{-\delta''}))$$

for some $\delta'' > 0$. Hence $P(\mathcal{E}_n) > 1 - a'' \exp(-n^{\delta''})$ for some $a'', \delta'' > 0$. \square

Lemma 23 *For M an arbitrary rooted loopless map with n edges satisfying \mathcal{E}_n^* , let*

$$\mu_M := \frac{1}{n} \sum_{e \in E} \delta_{d(e)/(4n/3)^{1/4}}, \quad \mu_B := \frac{1}{|E_B|} \sum_{e \in E_B} \delta_{d_B(e)/(2|E_B|)^{1/4}}.$$

Then $W_1(\mu_M, \mu_B) = O(n^{-1/32})$, and $|\sup(\mu_M) - \sup(\mu_B)| = O(n^{-1/32})$.

Proof: Let

$$\tilde{\mu}_M := \frac{1}{n} \sum_{e \in E} \delta_{d_B(e)/(4n/3)^{1/4}}, \quad \tilde{\mu}_B := \frac{1}{|E_B|} \sum_{e \in E_B} \delta_{d_B(e)/(4n/3)^{1/4}},$$

and let $F_{\tilde{\mu}_M}(x)$ and $F_{\tilde{\mu}_B}(x)$ be the respective cumulative functions of $\tilde{\mu}_M$ and $\tilde{\mu}_B$. The property given by \mathcal{E}_n''' ensures that $|F_{\tilde{\mu}_M}(x) - F_{\tilde{\mu}_B}(x)| \leq n^{3/4}/n = n^{-1/4}$. Since $F_{\tilde{\mu}_M}(x) = F_{\tilde{\mu}_B}(x) = 1$ for $x \geq (3/4)^{1/4} \log(n)^2$ we conclude that $W_1(\tilde{\mu}_M, \tilde{\mu}_B) \leq (3/4)^{1/4} \log(n)^2 n^{-1/4}$. Moreover, since $|d(e) - d_B(e)| \leq 2n^{7/32}$, we have $W_1(\mu_M, \tilde{\mu}_M) \leq 2n^{7/32}/(4n/3)^{1/4} \leq 2n^{-1/32}$. Finally, since $|E_B|^{-1/4} = (4n/3)^{-1/4} \cdot (1 + O(n^{-1/4}))$, we have $W_1(\mu_B, \tilde{\mu}_B) = O(n^{-1/4})$. To conclude, we have $W_1(\mu_M, \mu_B) \leq W_1(\mu_M, \tilde{\mu}_M) + W_1(\tilde{\mu}_M, \tilde{\mu}_B) + W_1(\tilde{\mu}_B, \mu_B) = O(n^{-1/32})$. By similar arguments one proves $|\sup(\mu_M) - \sup(\mu_B)| = O(n^{-1/32})$. \square

This lemma, Theorem 2, and Claim 3 then imply:

Theorem 24 *Let π_n be the distance-profile of the random rooted loopless map with n edges. Then $\mu_{4/3}(\pi_n)$ satisfies the ISE limit property.*

One can also easily prove the analogue of Proposition 20 for random rooted loopless maps:

Proposition 25 *Let R_n be the radius of the random rooted loopless map G_n with n edges. Then $R_n/(4n/3)^{1/4}$ converges in law to the width of μ_{ISE} , and the convergence also holds for the moments.*

Proof: The convergence in law to the width of ISE directly follows from Theorem 24. It remains to show that the latter convergence also holds for the moments. First note that $P(\neg \mathcal{E}_n^*) = O(\exp(-\Omega(\log(n)^2)))$, which is $o(n^{-k})$ for all $k \geq 1$. Hence for computing the moments of $r(G_n)$ we can condition on \mathcal{E}_n^* . Moreover conditioning on \mathcal{E}_n^* we have $r(G_n) \leq \text{Diam}(B) + 2n^{7/32}$, hence $r(G_n)/n^{1/4}$ has the uniform exponential decay property, hence the convergence of the moments holds. \square

5.2 Profile of random rooted maps

Very similarly as for loopless maps, a rooted map decomposes (along loops) as a tree of components that are loopless maps. All the arguments used in Section 5.1 can be recycled here (starting with the result proved in Banderier et al. (2001) that the random rooted map with n edges has almost surely a “giant” loopless component whose size is concentrated around $2n/3$). We therefore obtain.

Theorem 26 *Let G_n be the uniformly random rooted map with n edges, let π_n be its distance-profile and R_n be its radius. Then $\mu_{8/9}(\pi_n)$ satisfies the ISE limit property. Moreover $R_n/(8n/9)^{1/4}$ converges in law to the width of μ_{ISE} , and the convergence also holds for the moments.*

Remark 27 *Theorem 26 can alternatively be recovered from the results in Chassaing and Schaeffer (2004) for the profile and radius of rooted random quadrangulations, combined with the recent bijection in Ambjørn and Budd (2013) (which preserves the profile).*

Remark 28 *We have shown in this section that the ISE limit property for random rooted simple maps implies the ISE limit property for random rooted maps (via random rooted loopless maps). In contrast, we do not know how to prove that the ISE limit property for random rooted maps implies the ISE limit property for random rooted simple maps.*

6 Conclusion

Regarding the *typical distance* to the root, let d_n denote the distance to the root-vertex of a random edge in a random rooted simple (resp. loopless, general) map with n edges. The results we have obtained imply the following: the random variable $d_n/(an)^{1/4}$ (with $a = 2$ for simple maps, $a = 4/3$ for loopless maps, and $a = 8/9$ for general maps) converges to the random variable that gives the expectation of the nonnegative shift of μ_{ISE} (which is known to be distributed as $\text{sup}(\mu_{\text{ISE}})$), whose cumulative function is

$$\varphi(r) = \frac{4}{\sqrt{\pi}} \int_0^\infty d\xi \xi^2 e^{-\xi^2} \left(1 - 6 \frac{1 - \cosh(\tilde{r}\sqrt{\xi}) \cos(\tilde{r}\sqrt{\xi})}{(\cosh(\tilde{r}\sqrt{\xi}) - \cos(\tilde{r}\sqrt{\xi}))^2} \right), \quad \text{with } \tilde{r} = 2^{3/4}r.$$

Regarding the conjectural convergence of the random rooted plane graph with n edges to the Brownian map, the bijection of Bernardi et al. (2014) on which the present work relies can be composed with a bijection (of the “Ambjørn-Budd type”) given in Bouttier et al. (2013) between rooted eulerian triangulations and rooted bipartite maps, in a way that preserves the profile. This gives thus a bijective coupling of rooted (outer-triangular) plane graphs with rooted bipartite maps; and it is tempting to conjecture that the Gromov-Hausdorff distance between the metric spaces (rescaled by $n^{1/4}$) of coupled maps converges to 0 in probability. This (and the fact that a random rooted plane graph has almost surely a “giant” outer-triangular component, as shown in Lemma 18) would solve the problem, since the random rooted bipartite map with n edges has been recently shown in Abraham (2013) to converge to the Brownian map.

Another perspective is to study the distances in classes of random *non-embedded planar graphs*. Apart from the case of triangulations treated in Addario-Berry and Albenque (2013) much less is known for these models. On the distance profile, the most precise result known at the moment, shown by Chapuy et al. (2010), is that the diameter is $n^{1/4+o(1)}$ in probability.

Acknowledgements

The authors thank Marie Albenque, Nicolas Broutin, Lucas Gérin, Marc Noy and Gilles Schaeffer for interesting discussions, and Grégory Miermont for pointing to the reference Le Gall (2006).

References

- C. Abraham. Rescaled bipartite planar maps converge to the brownian map, 2013. In preparation.
- L. Addario-Berry and M. Albenque. The scaling limit of random simple triangulations and random simple quadrangulations, 2013. arXiv:1306.5227.
- D. J. Aldous. Tree-based models for random distribution of mass. pages 625–641, 1993.
- J. Ambjørn and T. Budd. Trees and spatial topology change in causal dynamical triangulations. *J. Phys. A: Math. Theor.*, 46(31):315201, 2013.
- C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Random maps, coalescing saddles, singularity analysis, and Airy phenomena. *Random Structures Algorithms*, 19(3/4):194–246, 2001.
- O. Bernardi and E. Fusy. Unified bijections for maps with prescribed degrees and girth. *J. Combin. Theory Ser. A*, 119(6):1351–1387, 2012.
- O. Bernardi, G. Collet, and E. Fusy. A bijection for plane graphs and its applications. In *Proceedings of the 11th Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, pages 52–61, 2014.
- M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(337-368), 2000.
- J. Bouttier, P. D. Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1), 2004.
- J. Bouttier, E. Fusy, and E. Guitter. On the two-point function of general planar maps and hypermaps, 2013. arXiv:1312.0502.
- G. Chapuy, E. Fusy, O. Giménez, and M. Noy. On the diameter of random planar graphs. In *DMTCS Proceedings, 21st International Meeting on Probabilistic, Combinatorial, and Asymptotic Methods in the Analysis of Algorithms (AofA'10)*, 2010.
- P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.
- P. Flajolet and R. Sedgewick. *Analytic combinatorics*. Cambridge University Press, 2009.

- J. Gao and N. Wormald. The size of the largest components in random planar maps. *SIAM J. Discrete Math.*, 12(2):217–228, 1999.
- J. F. Le Gall. A conditional limit theorem for tree-indexed random walk. *Stochastic Process. Appl.*, 116: 539–567, 2006.
- J. F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3): 621–670, 2007.
- J. F. Le Gall. Uniqueness and universality of the brownian map. *Ann. Probab.*, 41:2880–2960, 2013.
- J. F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.
- G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210:319–401, 2013.
- G. Miermont and M. Weill. Radius and profile of random planar maps with faces of arbitrary degrees. *Electron. J. Probab.*, 13:79–106, 2008.
- W. Schnyder. Planar graphs and poset dimension. *Order*, 5:323–343, 1989.

