DEFORMATIONS OF THE BRAID ARRANGEMENT AND TREES
DEDICATED TO IRA GESSEL FOR HIS RETIREMENT

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Abstract. We establish counting formulas and bijections for deformations of the braid arrangement. Precisely, we consider real hyperplane arrangements such that all the hyperplanes are of the form \( x_i - x_j = s \) for some integer \( s \). Classical examples include the braid, Catalan, Shi, semiorder and Linial arrangements, as well as graphical arrangements. We express the number of regions of any such arrangement as a signed count of decorated plane trees. The characteristic and coboundary polynomials of these arrangements also have simple expressions in terms of these trees.

We then focus on certain “well-behaved” deformations of the braid arrangement that we call transitive. This includes the Catalan, Shi, semiorder and Linial arrangements, as well as many other arrangements appearing in the literature. For any transitive deformation of the braid arrangement we establish a simple bijection between regions of the arrangement and a set of plane trees defined by local conditions. This answers a question of Gessel.

1. Introduction

We consider real hyperplane arrangements made of a finite number of hyperplanes of the form

\[ H_{i,j,s} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = s \}, \]

with \( i, j \in \{1, \ldots, n\} \) and \( s \in \mathbb{Z} \). In particular, given an integer \( n \) and a finite set of integers \( S \), the \( S \)-braid arrangement in dimension \( n \), denoted \( A_S(n) \), is the arrangement made of the hyperplanes \( H_{i,j,s} \) for all \( 1 \leq i < j \leq n \) and all \( s \in S \). Classical examples include the braid, Catalan, Shi, semiorder, and Linial arrangements, which correspond to \( S = \{0\} \), \( \{-1,0,1\} \), \( \{0,1\} \), \( \{-1,1\} \), and \( \{1\} \) respectively. These arrangements are represented in Figure 1. We refer the reader to [28] for an introduction to the general theory of hyperplane arrangements.

There is an extensive literature on counting regions of deformations of the braid arrangement. Seminal results were established by Stanley [30, 27], Postnikov and Stanley [25], and Athanasiadis [4]; see Section 2.3 for additional references. It has been observed by Ira Gessel that for the five classical arrangements defined above (and some generalizations thereof), the regions are equinumerous to some simple families of binary trees [15, 13]. Our goal is to show that this is not a coincidence, and provide a unified

Date: June 28, 2017.

This work was partially supported by NSF grant DMS-1400859. Part of it was completed while visiting MIT in the Spring 2016. Many thanks to Ira Gessel for suggesting this problem and providing many valuable inputs, and to Sam Hopkins and Alexander Postnikov for interesting discussions.
Our results come in two degrees of generality: results for general (integer) deformations of the braid arrangement, and results for transitive deformations.

For general deformations of the braid arrangement, we express the number of regions as a signed count of some (decorated, labeled, k-ary) trees with n nodes, that we call boxed trees (see Theorems 3.4 and 4.2). More generally, we express the characteristic polynomial and coboundary polynomial (equivalently, Tutte polynomial) of these arrangements in terms of boxed trees (see Theorem 5.2).

We then focus on certain deformations that we call transitive. This is a condition on the triples \((i, j, s)\) indexing the hyperplanes (see Definition 4.3). Examples include the S-braid arrangements for \(S \subseteq \{-1,0,1\}\), or \(S\) an interval containing 1, or \(S\) such that \([-k..k] \subseteq S \subseteq [-2k..2k]\) for some \(k\), and the \(G\)-Shi arrangement for any graph \(G\). For transitive deformations, we show that the regions of the arrangement are equinumerous to a set of (labeled, plane) trees satisfying simple local rules (see Theorem 3.8 and 4.6).

Furthermore, we establish a simple bijection between the regions and the trees (see Example 1.1). We now illustrate our bijective results on an example.

**Example 1.1.** The regions of the Linial arrangement \(A_{\{1\}}(n)\) are in bijection with the set \(T_{\{1\}}(n)\) of binary trees with \(n\) labeled node such that: for all node \(u \in [n]\) if at least one of the children of \(u\) is a node, then the rightmost node \(v\) satisfies \(v < u\). See Figure 2 for the case \(n = 3\).

The bijection associates to any tree \(T \in T_{\{1\}}(n)\), the region made of the points \((x_1, \ldots, x_n)\) satisfying the following inequalities for all \(1 \leq i < j \leq n\): \(x_i - x_j < 1\) if and only if either \(\text{drift}(i) \leq \text{drift}(j)\) or \(\text{drift}(i) = \text{drift}(j) + 1\) and \(i\) appears before \(j\) in the postfix order of \(T\), where \(\text{drift}(v)\) is the number of ancestors of \(v\) (including \(v\)) which are right-children.

In the case of the Catalan arrangement \(A_{\{-1,0,1\}}(n)\) (and the generalization \(A_{[-m..m]}(n)\)) our bijection builds on a classical construction. In the case of the Shi arrangement \(A_{\{0,1\}}(n)\) (and the generalization \(A_{[-m+1..m]}(n)\)) our bijection is a close relative to a bijection of Athanasiadis and Linusson [7]; see Section 9.1. But already in the case \(A_{\{1\}}(n)\), no bijection was known, although several sets of trees where known to be equinumerous to the regions of \(A_{\{1\}}(n)\) [25, 24, 23]. We give a direct proof of our bijections in the case of the Catalan, Shi, semiorder and Linial arrangements, while in the general case we use the counting formula established by other means.

The paper is organized as follows. In Section 2, we give basic definitions about trees and hyperplane arrangements, and recall some known results. In Section 3, we give...
formulas for the number of regions in S-braid arrangements. In Section 4 we extend these results to general deformations of the braid arrangement. In Section 5 we give formulas for the characteristic and coboundary polynomials of these arrangements. In Section 6 we compute the generating functions for the number of regions, and coboundary polynomials of deformed braid arrangements. In Section 7 we prove all our counting formulas. In Section 8 we establish our bijections. We conclude in Section 9 with some remarks and open questions.

2. Definitions and known results

2.1. Basic definitions. A real hyperplane arrangement in dimension n is a set A of affine hyperplanes of \( \mathbb{R}^n \). For instance, the braid arrangement \( A_{\{0\}}(n) \) is the set \( \{ H_{i,j,0} \}_{1 \leq i < j \leq n} \) of \( \binom{n}{2} \) hyperplanes. The regions of A are connected components of \( \mathbb{R}^n \setminus \bigcup_{H \in A} H \). We denote by \( r_A \) the number of regions. For instance, it is easy to see that \( r_{A_{\{0\}}(n)} = n! \). We refer the reader to [28] for an introduction to the general theory of hyperplane arrangements.

We denote \( \mathbb{N} = \{0, 1, 2, \ldots \} \). For \( a, b \in \mathbb{Z} \), we denote \( [a..b] = \{ i \in \mathbb{Z} \mid a \leq i \leq b \} \), and \([b] = [1..b] \). For a set \( S \), we denote \( |S| \) the cardinality. For a ring \( R \), we denote \( R[t] \) and \( R[[t]] \) respectively the set of polynomials and formal power series in \( t \) with coefficients in \( R \). We extend the notation to several variables so that \( R[y][[t_1, t_2]] \) is the set of formal power series in \( t_1, t_2 \) with coefficients in \( R[y] \). For \( G(t) \in R[[t]] \), we denote \( [t^k]G(t) \) the coefficient of \( t^k \) in \( G \).
2.2. Labeled plane trees. A tree is a finite connected acyclic graph. A rooted plane tree is a tree with a vertex distinguished as the root, together with an ordering of the children of each vertex (we think of the vertices as being ordered from left to right). A vertex in a rooted plane tree is a leaf if it has no children, and a node otherwise. We denote by $\mathcal{T}$ the set of rooted plane trees with labeled nodes (if the tree has $n$ nodes, then the nodes have distinct labels in $[n]$, while the leaves are not labeled). We denote $\mathcal{T}^{(m)}$ the set of $(m+1)$-ary trees in $\mathcal{T}$ (i.e. the trees such that every node has $m+1$ children). We also denote $\mathcal{T}^{(m)}(n)$ the set of trees with $n$ nodes in $\mathcal{T}^{(m)}$. A tree in $\mathcal{T}^{(2)}(13)$ is represented in Figure 3(a). For a non-root node $v$ of $T \in \mathcal{T}$, we denote $\text{parent}(v)$ the parent of $v$, and $\text{lsib}(v)$ the number of left-siblings of $v$ (including leaves).

We compare nodes of $\mathcal{T}$ according to their labels, so that $u < v$ means that the label of $u$ is less than the label of $v$.

![Figure 3](image)

Figure 3. (a) A (rooted plane node-labeled) tree in $\mathcal{T}^{(2)}(13)$. (b) A $S$-boxed tree for $S = \{-1, 2\}$ (note for instance that $-1 \in S$ imposes that if a box contains both a node $u$ and its middle child $v$, then $u > v$).

2.3. Known counting results about deformed braid arrangements. It has been observed that for all set $S \subseteq \{-1, 0, 1\}$, the regions in $S$-braid arrangements are equinumerous to certain family of trees in $\mathcal{T}^{(1)}$. Up to symmetry, we only need to consider the braid, Catalan, Shi, semiorder, and Linial arrangements, which are represented in Figure 1. We now describe the corresponding families of trees (see Figure 4). We call left node (resp. right node) of a tree $T \in \mathcal{T}^{(1)}$ a non-root node $v$ such that $\text{lsib}(v) = 0$ (resp. $\text{lsib}(v) = 1$).

- The regions of the Catalan arrangement $\mathcal{A}_{\{-1,0,1\}}$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$. Clearly, there are $n!\text{Cat}(n)! = \frac{(2n)!}{(n+1)!}$ such trees (because there are $\text{Cat}(n)$ binary trees with $n$ nodes and $n!$ ways of labeling their nodes).

- The regions of the Shi arrangement $\mathcal{A}_{\{0,1\}}$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$ such that
  (i) for every right node $v$, $\text{parent}(v) > v$.
  It is easy to see that there are $(n+1)^{n-1}$ such trees because they are in bijection with Cayley rooted forests with $n$ vertices.

- The regions of the semiorder arrangement $\mathcal{A}_{\{-1,1\}}$ are equinumerous to the trees in $\mathcal{T}^{(1)}(n)$ such that
  (ii) for every left node $v$, if the right-sibling of $v$ is a leaf then $\text{parent}(v) > v$. 
The regions of the Linial arrangement $A_{\{1\}}$ are equinumerous to the trees in $T^{(1)}(n)$ such that
(iii) for every left node $v$, parent$(v) > v$, and for every right node $v$, parent$(v) < v$.

Additionally, the braid arrangement $A_{\{0\}}(n)$ has $n!$ regions, which is the number of trees in $T^{(1)}(n)$ such that the label of every non-root node is greater than the label of its parent. Gessel had enumerated trees in $T^{(1)}(n)$ according to the number of left and right ascents and descents (the original proof was not published, but a different proof was given in [18]). He observed the above identities between the number of regions and the number of trees, and raised the question of finding a uniform, possibly bijective, explanation of these coincidences [13]. We provide such an explanation in this paper.

![Figure 4](image-url)

**Figure 4.** The conditions (i), (ii), (iii) appearing in the literature for the classes of trees equinumerous to the regions of the Shi, semiorder and Linial arrangements. The characterization (iii') proved in this paper for the Linial arrangement appears to be new (see Section 9.2 for a bijection between (iii) and (iii')). Nodes are represented by labeled discs, while leaves are represented by black dots (here, the nature of some vertices is left unspecified).

The above results have natural generalizations. The $m$-Catalan arrangement $A_{[-m..m]}(n)$ has $\left(\frac{(m+1)n!}{(mn+1)}\right)$ regions, which is the number of trees in $T^{(m)}(n)$. A simple bijective proof of this well-known result (see e.g. [30, Section 4]) is recalled in Section 8.1. The $m$-Shi arrangement $A_{[-m+1..m]}(n)$ has $(mn+1)^{n-1}$ regions, which is the number of $m$-parking functions of size $n$. This result was first established by Shi [26], and at least two bijective proofs of this fact are known [7, 27], beside non bijective proofs (see [25, 11, 5]). We discuss these bijections further in Section 9.1. Note that $(mn+1)^{n-1}$ is also the number of trees in $T^{(m)}(n)$ such that if a node $v$ is the right-most child of a node $u$, then $u > v$ (this generalizes (i)). The result about the semiorder arrangement is equivalent to a result of Chandon [10]. More generally, the $m$-semiorder arrangement $A_{[-m..m]\{0\}}(n)$ has regions equinumerous to trees in $T^{(m)}(n)$ such that if a node $v$ is the leftmost child of a node $u$, and all its siblings are leaves, then $u > v$ (this generalizes (ii)). This fact is easily deduced from the generating function equation given in [25, Theorem 7.1]. Lastly, the case of the Linial arrangement was conjectured by Linial and Ravid, and proved in [25] and independently in [4] by equating two generating functions. Actually, the $m$-Linial arrangement $A_{[-m+1..m]\{0\}}(n)$ also has regions equinumerous to a subset of $T^{(m)}(n)$, although this interpretation differs from (iii) and seems to have gone unnoticed. Namely, the regions of $A_{[-m+1..m]\{0\}}(n)$ are equinumerous to the trees in...
$T^{(m)}(n)$ satisfying both generalizations of the conditions (i) and (ii). This characterization is represented by Condition (iii') in Figure 4. The fact that the family of trees associated to the intersection $A_{[-m+1..m]\{0\}}(n) = A_{[-m+1..m]}(n) \cap A_{[-m..m]\{0\}}(n)$ is the intersection of the family of trees associated to $A_{[-m+1..m]}(n)$ and $A_{[-m..m]\{0\}}(n)$ is a general feature of the theory developed in the present paper (see Remark 3.10). A bijective link between Conditions (iii) and (iii') is also given in Section 9.2.

Although we cannot give an exhaustive bibliography about the deformations of the braid arrangement, we should mention a few additional references which are relevant to the present article. Formulas for the number of regions of several additional deformations of the braid arrangements (for instance $A_{[-\ell..m]}(n)$ for $\ell \geq -1$) are given in [25]. The characteristic and coboundary polynomials of some of the arrangements above have been computed in [5, 1]. In a different direction, several deformations of the braid arrangements associated to a graph $G = ([n], E)$ have been considered in the literature. The most classical is the $G$-graphical arrangement made of the hyperplanes $H_{i,j,0}$ for all $i, j \in E$. Another important example is the $G$-Shi arrangement considered for instance in [3, 7]. This arrangement is made of the hyperplanes $H_{i,j,0}$ for all $i, j \in [n]$, and $H_{i,j,1}$ for all $i, j \in E$ with $i < j$ (so that it is the braid arrangement if $G$ has no edge, and the Shi arrangement if $G = K_n$). In yet another direction, several authors have considered deformed braid arrangements with hyperplanes $H_{i,j,s}$ for generic, non-integer values of $s$ (see e.g. [25, 27, 28, 17]), but we will not consider such situations here.

3. Counting regions of $S$-braid arrangements

In this section we present our counting results for the regions of $S$-braid arrangements. Throughout this section, $S$ is a finite set of integers, $m = \max(|s|, s \in S)$, and $n$ is a non-negative integer.

We start with the definition of $S$-boxed trees, and express the number of regions of $A_S(n)$ as a signed count of $S$-boxed trees. Then we go on to express the number of regions of $A_S(n)$ as an unsigned count of trees in the case where $S$ is transitive.

**Definition 3.1.** Let $T$ be a tree in $T$, and let $u$ be a node. If one of the children of $u$ is a node, we call cadet-node of $u$, and denote cadet($u$), the rightmost such child.

- A cadet sequence is a non-empty sequence $(v_1, \ldots, v_k)$ of nodes such that for all $i$ in $[k-1]$, $v_{i+1} = \text{cadet}(v_i)$.
- A $S$-cadet sequence is a cadet sequence $(v_1, \ldots, v_k)$ such that for all $1 \leq i < j \leq k$, if $\sum_{p=i+1}^{j} \text{lsib}(v_p) \in S \cup \{0\}$ then $v_i < v_j$, and if $-\sum_{p=i+1}^{j} \text{lsib}(v_p) \in S$ then $v_i > v_j$.

Note that a $S$-cadet sequence $(v_1, \ldots, v_k)$ of $T \in T^{(m)}$ satisfies in particular \text{lsib}(v_j) \in [0..m] \setminus \{s \in S \mid -s \in S\}$ for all $j \in [2..k]$.

**Example 3.2.** Let $T \in T^{(m)}$.

- For $S = [-m..m]$, the $S$-cadet sequences of $T$ contain a single vertex.
\begin{itemize}
  \item For $S = [-\ell, m]$ with $0 \leq \ell \leq m$, the $S$-cadet sequences of $T$ are the cadet sequences $(v_1, \ldots, v_k)$ satisfying $v_1 < v_2 < \cdots < v_k$ and $\text{lsib}(v_p) \in [\ell + 1, m]$ for all $p \in [2..k]$.
  \item For $S = [-\ell, m] \setminus \{0\}$ with $0 \leq \ell \leq m$, the $S$-cadet sequences of $T$ are the cadet sequences $(v_1, \ldots, v_k)$ satisfying $v_1 < v_2 < \cdots < v_k$ and $\text{lsib}(v_p) \in \{0\} \cup [\ell + 1, m]$ for all $p \in [2..k]$.
  \item For $S = \{-2, 0, 1, 2\}$, the $S$-cadet sequences of $T$ have size at most 2, and the $S$-cadet sequences of size 2 are of the form $(v_1, v_2)$ with $\text{lsib}(v_2) = 1$ and $v_1 < v_2$.
\end{itemize}

\textbf{Definition 3.3.} A boxed tree is a pair $(T, B)$, where $T$ is in $T$, and $B$ is a set of boxed sequences of $T$ containing every node exactly once. It is a $S$-boxed tree if $T \in T^{(m)}$, and $B$ contains only $S$-cadet sequences. We denote $U_S(n)$ the set of $S$-boxed trees with $n$ nodes.

We represent boxed trees as trees decorated with boxes partitioning the nodes into cadet sequences, as in Figure 3(b).

\textbf{Theorem 3.4.} Let $S$ be a finite set of integers and $n$ be a positive integer. The number of regions of the hyperplane arrangement $A_S(n)$ is

\begin{equation}
  r_S(n) = \sum_{(T, B) \in U_S(n)} (-1)^{|n - |B|}.
\end{equation}

The proof of Theorem 3.4 is delayed to Section 7. We will now give a simpler expression for $r_S(n)$ in the cases where the set $S$ is “well behaved”.

\textbf{Definition 3.5.} A set $S$ of integers is said transitive if

\begin{itemize}
  \item $\forall s, t \notin S$ such that $st > 0$, $s + t \notin S$,
  \item $\forall s, t \notin S$ such that $s > 0$ and $t \leq 0$, $s - t \notin S$ and $t - s \notin S$.
\end{itemize}

\textbf{Example 3.6.} \begin{itemize}
  \item All the subsets of $\{-1, 0, 1\}$ are transitive.
  \item All the intervals of integers containing 1 are transitive.
  \item Sets of the form $S = I \setminus k\mathbb{Z}$, where $I$ is an interval containing 1 are transitive.
  \item Sets $S$ such that $[-m/2, \ldots, m/2] \subseteq S \subseteq [-m..m]$ for some $m$ are transitive.
  \item A set $S$ such that $\{-s, s \in S\} = S$ is transitive if and only if the set of positive integers not in $S$ is closed under addition.
  \item A set $S$ such that $[m] \subseteq S \subseteq [-m..m]$ is transitive if and only if the set of negative integers not in $S$ is closed under addition.
\end{itemize}

\textbf{Definition 3.7.} We denote $T_S(n)$ the set of trees $T$ in $T^{(m)}(n)$ such that any nodes $u, v$ such that cadet($u$) = $v$ satisfies the following:

\textbf{Condition(S):} if $\text{lsib}(v) \notin S \cup \{0\}$ then $u < v$, and if $-\text{lsib}(v) \notin S$ then $u > v$.

\textbf{Theorem 3.8.} If $S$ is transitive, then regions of the hyperplane arrangement $A_S(n)$ are equinumerous to the trees in $T_S(n)$.

\textbf{Example 3.9.} \begin{itemize}
  \item $T_{[-m..m]}(n) = T^{(m)}(n)$.
  \item $T_{[-m+1..m]}(n)$ is the set of trees in $T^{(m)}(n)$, such that any non-root node having no right-sibling (not even leaves) is less than its parent.
\end{itemize}
• \(T_{[m]}(n)\) is the set of trees in \(T^{(m)}(n)\), such that such that any cadet-node \(v\) is less than its parent.

• More generally, for \(0 \leq \ell \leq m\), \(T_{[-\ell,m]}(n)\) is the set of trees in \(T^{(m)}(n)\), such that any cadet-node \(v\) having more than \(\ell\) left-sibling is less than its parent.

And \(T_{[-\ell,m]}\setminus\{0\}(n)\) is the set of trees in \(T^{(m)}(n)\), such that any cadet-node \(v\) having neither left sibling or more than \(\ell\) left-siblings is less than its parent.

Remark 3.10. For any sets \(S, S' \subset \mathbb{Z}\), \(T_S(n) \cap T_{S'}(n) = T_{S \cap S'}(n)\). For instance the set \(T_{[1]}(n)\) of trees associated to Linial arrangement, is the intersection of the set of trees \(T_{[0,1]}(n)\) associated to the Shi arrangement, and the set of trees \(T_{[-1,1]}(n)\) associated to the semiorder arrangement.

Moreover, each element \(s \in [-m..m] \setminus S\) gives a simple condition for trees in \(T_S(n)\): for \(s > 0\) the condition is that a cadet node with \(s\) left siblings is greater than its parent, while for \(s \leq 0\) the condition is that a cadet node with \(-s\) left siblings is less than its parent. This is represented in Figure 5.

<table>
<thead>
<tr>
<th>Condition for (s \in [m] \setminus S)</th>
<th>Condition for (-s \in [-m..0] \setminus S)</th>
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<tbody>
<tr>
<td>(s) (\Rightarrow u &lt; v)</td>
<td>(s) (\Rightarrow u &gt; v)</td>
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Figure 5. Conditions for trees to be in \(T_S(n)\). Each element \(s \in [-m..m] \setminus S\) imposes one condition.

Theorem 3.8 is an easy consequence of Theorem 3.4 and the following lemma.

Lemma 3.11. If \(S\) transitive, a cadet sequence \((v_1, \ldots, v_k)\) is a \(S\)-cadet sequence if and only if for all \(i \in [k - 1]\),

\((*)\) if \(\text{lsib}(v_{i+1}) \in S \cup \{0\}\) then \(v_i < v_{i+1}\), and if \(-\text{lsib}(v_{i+1}) \in S\) then \(v_i > v_{i+1}\).

Proof. It is clear that the condition \((*)\) is necessary. We now prove that it is sufficient, by induction on \(k\). The case \(k = 1\) is trivial. Now suppose that \(k > 1\) and \(\gamma = (v_1, \ldots, v_k)\) satisfies \((*)\). Since \(\gamma' = (v_1, \ldots, v_{k-1})\) satisfies \((*)\), it is a \(S\)-cadet sequence. Hence we only need to check that for all \(i \in [k - 1]\),

\((**)\) if \(\sum_{p=i+1}^{k} \text{lsib}(v_p) \in S \cup \{0\}\) then \(v_i < v_k\), and if \(-\sum_{p=i+1}^{k} \text{lsib}(v_p) \in S\) then \(v_i > v_k\).

The case \(i = k - 1\) of \((**)^{\prime}\) is directly given by \((*)\). We now consider \(i \in [k - 2]\), and consider several cases. Suppose first that \(v_i < v_{k-1} < v_k\). In this case, \(-\sum_{p=i+1}^{k-1} \text{lsib}(v_p) \notin S\) (since \(\gamma'\) is a \(S\)-cadet sequence), \(-\text{lsib}(v_k) \notin S\) (since \(\gamma\) satisfies \((*)\)), hence \(-\sum_{p=i+1}^{k} \text{lsib}(v_p) \notin S\) (since \(S\) is transitive), hence Condition \((**)^{\prime}\) holds for \(i\). The case \(v_i > v_{k-1} > v_k\) is treated similarly. Suppose next that \(v_i > v_{k-1} < v_k\). In this case, \(\sum_{p=i+1}^{k-1} \text{lsib}(v_p) \notin S \cup \{0\}\) (since \(\gamma'\) is a \(S\)-cadet sequence), \(-\text{lsib}(k) \notin S\) (since \(\gamma\) satisfies \((*)\)), hence \(\sum_{p=i+1}^{k} \text{lsib}(v_p) \notin S \cup \{0\}\) and \(-\sum_{p=i+1}^{k} \text{lsib}(v_p) \notin S\) (since \(S\) is transitive), hence Condition \((**)^{\prime}\) holds for \(i\). The case \(v_i < v_{k-1} > v_k\) is treated similarly. Thus Condition \((**)^{\prime}\) holds for all \(i\), and \(\gamma\) is a \(S\)-cadet sequence. \(\square\)
Proof of Theorem 3.8. Let $T \in \mathcal{T}^{(m)}(n)$, and let $v = \text{cadet}(u)$. We claim that that $u$ and $v$ can be in the same box of a $S$-boxed tree $(T, B)$ if and only if $u$ and $v$ do not satisfy Condition($S$) of Definition 3.7. Indeed, by Lemma 3.11 in the case $u < v$ (resp. $u > v$) the vertices $u$ and $v$ can be in the same box if and only if $-\text{lsub}(v) \notin S$ (resp. $\text{lsub}(v) \notin S \cup \{0\}$), and this holds if and only if Condition($S$) does not hold.

For a tree $T$ in $\mathcal{T}^{(m)}(n)$, we denote $\mathcal{B}_T = \{B \mid (T, B) \in \mathcal{U}_S(n)\}$. By Theorem 3.4:

\begin{equation}
(3.2) \quad r_S(n) = \sum_{T \in \mathcal{T}(n)} \sum_{B \in \mathcal{B}_T} (-1)^{|n|-|B|} + \sum_{T \in \mathcal{T}^{(m)}(n) \setminus \mathcal{T}(n)} \sum_{B \in \mathcal{B}_T} (-1)^{|n|-|B|}.
\end{equation}

By the above claim, for all $T$ in $\mathcal{T}_S$, $\mathcal{B}_T$ contain a single element because every node of $T$ must be in a different box. Thus the first sum of (3.2) contributes $|\mathcal{T}_S(n)|$. We now prove that the second sum is 0 using a sign reversing involution. For a tree $T \in \mathcal{T}^{(m)}(n) \setminus \mathcal{T}(n)$, we pick the smallest vertex $v = \text{cadet}(u)$ such that Condition($S$) does not hold, and define an involution $\varphi$ on $\mathcal{B}_T$ as follows:

- if $u$ and $v$ are in the same box of $B$, then $\varphi(B)$ is obtained by splitting the box containing them into $u$ and $v$,
- if $u$ and $v$ are in different boxes of $B$, then $\varphi(B)$ is obtained by merging these boxes.

Lemma 3.11 ensures that $\varphi(B) \in \mathcal{B}_T$ in the second situation. Since $\varphi$ is an involution on $\mathcal{B}_T$ changing the number of boxes by $\pm 1$, we get $\sum_{B \in \mathcal{B}_T} (-1)^{|n|-|B|} = 0$. Hence the second sum in (3.2) contributes 0. \hfill \Box

4. General deformations of the braid arrangement

In this section we extend the results of Section 3 to general deformations of the braid arrangement. We fix a positive integer $N$ and a $\binom{N}{2}$-tuple of finite sets of integers $S = (S_{a,b})_{1 \leq a < b \leq N}$. We call $S$-braid arrangement, and denote $\mathcal{A}_S$, the arrangement in $\mathbb{R}^N$ made of the hyperplanes

$$H_{a,b,s} = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_a - x_b = s\},$$

for all $1 \leq a < b \leq N$ and all $s \in S_{a,b}$. Note that if $S_{a,b} = S$ for all $a, b$, then $\mathcal{A}_S = \mathcal{A}_S(N)$.

We will now extend Theorem 3.4 to $S$-braid arrangements. We denote $m = \max(|s|, s \in \cup S_{a,b})$. For $1 \leq a < b \leq N$, we denote $S_{b,a} = S_{a,b}$, and $S_{a,b}^+ = \{s \geq 0 \mid -s \in S_{a,b}\}$, and $S_{b,a}^- = \{s > 0 \mid s \in S_{a,b}\} \cup \{0\}$.

Definition 4.1. A cadet sequence $(v_1, \ldots, v_k)$ of $T \in \mathcal{T}^{(m)}(N)$ is a $S$-cadet sequence if for all $1 \leq i < j \leq k$, $\sum_{p=i+1}^{j} \text{lsub}(v_p) \notin S_{v_i,v_j}^{-}$. A $S$-boxed tree is a boxed tree $(T, B)$ with $T \in \mathcal{T}^{(m)}(N)$, and $B$ containing only $S$-cadet sequences. We denote $\mathcal{U}_S$ the set of $S$-boxed trees.

Theorem 4.2. The number of regions of the hyperplane arrangement $\mathcal{A}_S$ is

\begin{equation}
(4.1) \quad \tau_S = \sum_{(T,B) \in \mathcal{U}_S} (-1)^{|n|-|B|}.
\end{equation}
The condition $\sum_{p=1}^{j} \text{lsib}(v_p) \not\in S_{v_i,v_j}$ is equivalent to: if $\sum_{p=1}^{j} \text{lsib}(v_p) \in S_{v_i,v_j} \cup \{0\}$ then $v_i < v_j$, and if $-\sum_{p=1}^{j} \text{lsib}(v_p) \in S_{v_i,v_j}$ then $v_i > v_j$. In particular, if $S_{a,b} = S$ for all $a,b$, then $\mathcal{U}_S = \mathcal{U}_S(N)$. Hence Theorem 4.2 generalizes Theorem 3.4.

We now generalize Theorem 3.8 to S-braid arrangements.

**Definition 4.3.** The tuple $S$ is said transitive if for all distinct integers $a,b,c \in [N]$ the following condition holds: if $s \not\in S_{a,b}^{-}$ and $t \not\in S_{b,c}^{-}$, then $s + t \not\in S_{a,c}^{-}$.

It is easy to see that if $S_{a,b} = S$ for all $a,b \in [N]$, then $S$ is transitive if and only if $S$ is transitive.

**Example 4.4.** If for all $a,b \in [N]$, $[-\lfloor m/2 \rfloor .. \lceil m/2 \rceil] \subseteq S_{a,b} \subseteq [-m..m]$, then $S$ is transitive.

**Definition 4.5.** We denote $\mathcal{T}_S$ the set of trees $T$ in $\mathcal{T}^{(m)}(N)$ such that any pair of nodes $u,v$ such that cadet$(u) = v$ satisfies $\text{lsib}(v) \in S_{u,v}$.

**Theorem 4.6.** If $S = (S_{a,b})_{1 \leq a < b \leq N}$ is transitive, then the regions of $\mathcal{A}_S$ are equinumerous to the trees in $\mathcal{T}_S$.

**Remark 4.7.** The condition $\text{lsib}(v) \not\in S_{u,v}^{-}$ is equivalent to: if $\text{lsib}(v) \not\in S_{u,v} \cup \{0\}$ then $u < v$, and if $-\text{lsib}(v) \in S_{u,v}$ then $u > v$. In particular, if $S_{a,b} = S$ for all $a,b \in [N]$, then $\mathcal{T}_S = \mathcal{T}_S(N)$. Hence Theorem 4.2 generalizes Theorem 3.4.

**Example 4.8.** Let $G = ([N], E)$ be a graph and let $S$, $S'$ be two finite sets of integers. Let $G(S,S')$ be the tuple $S = (S_{a,b})_{1 \leq a < b \leq N}$ defined by $S_{a,b} = S$ if $\{a,b\} \in E$ and $S_{a,b} = S'$ otherwise. Several cases are represented in Figure 6

$(1)$ For $S = \{-1,0,1\}$ and $S' = \{0,1\}$, the tuple $S = G(S,S')$ is transitive for any graph $G$, and $\mathcal{T}_S$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node $v$ is the right child of $u$, then either $\{u,v\} \in E$ or $u > v$ (or both).

$(2)$ For $S = \{-1,0,1\}$ and $S' = \{0\}$, the tuple $S = G(S,S')$ is transitive for any graph $G$, and $\mathcal{T}_S$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node $v$ is the right child of $u$, then $\{u,v\} \in E$.

$(3)$ For $S = \{0,1\}$ and $S' = \{0\}$, the tuple $S = G(S,S')$ is transitive for any graph $G$, and $\mathcal{T}_S$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node $v$ is the right child of $u$, then $\{u,v\} \in E$ and $u > v$.

$(4)$ For $S = \{0,1\}$ and $S' = \{-1,0\}$, the tuple $S = G(S,S')$ is transitive for any graph $G$, and $\mathcal{T}_S$ is the set of trees in $\mathcal{T}^{(1)}(N)$ such that if a node $v$ is the right child of $u$, then either $\{u,v\} \in E$ and $u > v$ or $\{u,v\} \not\in E$ and $u < v$.

**Proof of Theorem 4.6.** Given Theorem 4.2 we only need to prove that if $S = (S_{a,b})_{1 \leq a < b \leq n}$ is transitive, then

$$\sum_{(T,B) \in \mathcal{U}_S} (-1)^{|B|} = |\mathcal{T}_S|.$$

The proof of (4.2) is almost identical to that of Theorem 3.8 except that Lemma 3.11 is replaced by the following claim: if $S$ is transitive, a cadet sequence $(v_1, \ldots, v_k)$ of...
**Figure 6.** Some transitive deformations of the braid arrangement. These arrangements have the form $A_G(S, S')$, where $G$ is the graph having vertex set $\{3\}$ and edges $\{1, 2\}$ and $\{1, 3\}$.

$T \in \mathcal{T}^{(m)}(N)$ is a $S$-cadet sequence if and only if for all $i \in [k-1]$, $\text{lsib}(v_{i+1}) \notin S_{v_i, v_{i+1}}^{-}$.

**Proof of the claim:** It is clear that $\text{lsib}(v_{i+1}) \notin S_{v_i, v_{i+1}}^{-}$ is necessary. We now prove that it is sufficient, by induction on $k$. The case $k = 1$ is trivial. Now suppose that $k > 1$, and $\gamma = (v_1, \ldots, v_k)$ is a cadet sequence such that for all $i \in [k-1]$, $\text{lsib}(v_{i+1}) \notin S_{v_i, v_{i+1}}^{-}$. We want to prove that $\gamma$ is a $S$-cadet sequence. By the induction hypothesis, $\gamma' = (v_1, \ldots, v_{k-1})$ is a $S$-cadet sequence, so we only need to prove that for all $i \in [k-1]$, $\sum_{p=i+1}^{k} \text{lsib}(v_p) \notin S_{v_i, v_k}^{-}$. This is true by hypothesis for $i = k - 1$. Moreover, for $i \in [k-1]$, $\sum_{p=i+1}^{k} \text{lsib}(v_p) \notin S_{v_{k-1}, v_k}^{-}$ (since $\gamma'$ is a $S$-cadet sequence), and $\text{lsib}(v_k) \notin S_{v_{k-1}, v_k}^{-}$ (by hypothesis), hence $\sum_{p=i+1}^{k} \text{lsib}(v_p) \notin S_{i, k}^{-}$ (since $S$ is transitive). Hence $\gamma$ is a $S$-cadet sequence. This proves the claim.

One can then define a sign reversing involution on $S$-boxed trees showing (4.2), exactly as in the proof of Theorem 3.8.

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5. **Characteristic and coboundary polynomials**

In this section we give expressions for the characteristic polynomial and coboundary polynomial of deformed braid arrangements.

For a hyperplane arrangement $A \subset \mathbb{R}^n$, we denote $\chi_A(q)$ the characteristic polynomial (i.e., chromatic polynomial) of $A$, and $P_A(q, y)$ its coboundary polynomial. Recall that these are defined by $\chi_A(q) = P_A(q, 0)$, and

$$P_A(q, y) = \sum_{B \subseteq A, \cap H \in B \neq \emptyset} q^{\dim(\cap H \in B)}(y - 1)^{|B|}.$$ 

The characteristic polynomial contains a lot of information about the arrangement $A$. In particular, by a result of Zaslavsky [33], the number of regions of $r_A$ and the number of relatively bounded$^1$ regions $b_A$ are evaluations of $\chi_A(q)$:

\begin{align}
  r_A &= (-1)^n\chi_A(-1), \\
  b_A &= (-1)^{\text{rank}(A)}\chi_A(1),
\end{align}

$^1$A region of $A$ is relatively bounded if its intersection with the subspace generated by the vectors normal to the hyperplanes of $A$ is bounded.
where \( \text{rank}(A) \) is the dimension of the vector space generated by the vectors normal to the hyperplanes of \( A \). The characteristic polynomial is also equivalent to the Poincaré polynomial of the cohomology ring of the complexification of \( A \); see [22]. The coboundary polynomial is equivalent to the Tutte polynomial \( T_A(x, y) \) of \( A \) (that is, the Tutte polynomial of the semi-matroid associated with \( A \), in the sense of Ardila [1][2]):

\[
T_A(x, y) = (y - 1)^{-\text{rank}(A)} P_A((x - 1)(y - 1), y).
\]

In order to express the coboundary polynomials of deformed braid arrangements in terms of boxed trees, we will consider arrangements of all dimensions. Let \( \hat{S} = (S_{a,b})_{1 \leq a \leq b \leq N} \) be a \( \binom{N+1}{2} \)-tuple of finite sets, let \( m = \max(|s|, s \in \cup S_{a,b}) \), and let \( n = (n_1, \ldots, n_N) \in \mathbb{N}^N \). We denote \( |n| = n_1 + \ldots + n_N \), and

\[
V(n) = \{(a, i) \mid a \in [N], i \in [n_a]\}.
\]

We endow \( V(n) \) with the lexicographical order, that is, we denote \((a, i) < (b, j)\) if either \( a < b \), or \( a = b \) and \( i < j \). For \( u = (a, i) \) and \( v = (b, j) \in V(n) \), we denote \( S_{u,v} = S_{v,u} = S_{a,b} \) and if \( u < v \) we denote \( S_{u,v} = \{s \geq 0 \mid -s \in S_{a,b}\} \) and \( S_{v,u} = \{s > 0 \mid s \in S_{a,b}\} \cup \{0\} \). Lastly, we define \( A_{\hat{S}}(n) \) as the arrangement in \( \mathbb{R}^{|n|} \) with hyperplanes

\[
H_{u,v,s} = \{(x_w)_{w \in V(n)} \mid x_u - x_v = s\},
\]

for all \( u < v \) in \( V(n) \) and all \( s \in S_{u,v} \).

Note that \( A_{\hat{S}}(n) \) identifies with the arrangement \( A_{\hat{S}(n)} \), where

\[
(5.2) \quad \hat{S}(n) = (S'_{u,v})_{1 \leq u < v \leq |n|}
\]

with \( S'_{u,v} = S_{a,b} \) for all \( u \in \left[1 + \sum_{i=1}^{a-1} n_i \ldots \sum_{i=1}^{a} n_i\right] \) and \( v \in \left[1 + \sum_{i=1}^{b-1} n_i \ldots \sum_{i=1}^{b} n_i\right] \). For instance, \( A_{\hat{S}}(1,1,\ldots,1) = A_{\hat{S}}, \) and \( A_{\hat{S}}(n_1,0,\ldots,0) = A_{S_{1,1}}(n_1) \). We now describe boxed trees related to the arrangement \( A_{\hat{S}}(n) \).

- We denote \( T^{(m)}(n) \) the set of rooted plane \((m+1)\)-ary trees with \(|n|\) nodes labeled with distinct labels in \( V(n) \).
- A cadet sequence \((v_1,\ldots,v_k)\) of \( T \in T^{(m)}(n) \) is admissible if \( v_i < v_{i+1} \) for all \( i \in [k-1] \) such that \( \text{lsib}(v_{i+1}) = 0 \). A boxed tree \((T,B)\) is admissible if all the sequences in \( B \) are admissible. We denote \( U^{(m)}(n) \) the set of admissible boxed trees \((T,B)\) with \( T \in T^{(m)}(n) \).
- The \( \hat{S} \)-energy of a cadet sequence \((v_1,\ldots,v_k)\) of \( T \) is the number of pairs \( \{i,j\} \) with \( 1 \leq i < j \leq k \), such that \( \sum_{p=i+1}^{j} \text{lsib}(v_p) \in S_{v_i,v_j}^c \). The \( \hat{S} \)-energy of a boxed tree \((T,B) \in U^{(m)}(n) \), denoted \( \text{energy}_{\hat{S}}(T,B) \), is the sum of the energies of the cadet sequences in \( B \).
- We denote by \( U_{\hat{S}}(n) \) the set of boxed trees \((T,B) \in U^{(m)}(n) \) such that \( \text{energy}_{\hat{S}}(T,B) = 0 \).

Note that any boxed tree in \( U_{\hat{S}}(n) \) is admissible. Moreover, \( U_{\hat{S}}(n_1,0,\ldots,0) = U_{S_{1,1}}(n_1) \), and \( U_{\hat{S}}(1,1,\ldots,1) = U_{S} \), where \( S = (S_{a,b})_{1 \leq a < b \leq N} \).

We denote \( T_{\hat{S}}(n) \) the set of trees \( T \) in \( T^{(m)}(n) \) such that any pair of nodes \( u,v \) such that \( \text{cadet}(u) = v \) satisfies \( \text{lsib}(v) \in S_{u,v}^c \). Note that \( T_{\hat{S}}(n_1,0,\ldots,0) = T_{S_{1,1}}(n_1) \), and
\( T_S(1,1,\ldots,1) = T_S \), where \( S = (S_{a,b})_{1 \leq a < b \leq N} \). We say that \( \hat{S} \) is multi-transitive if \( \hat{S}(n) \) is transitive (in the sense of Definition 4.3) for all \( n \in \mathbb{N}^N \). Note that for \( N = 1 \) \( \hat{S} \) is multi-transitive if and only if \( S_{1,1} \) is transitive.

**Example 5.1.** If for all \( a, b \in [N] \), \([-\lfloor m/2 \rfloor \ldots \lceil m/2 \rceil] \subseteq S_{a,b} \subseteq [-m..m] \), then \( \hat{S} \) is multi-transitive. Also, if \( S_{a,a} \) is transitive for all \( a \in [N] \), and \( S_{a,b} = [-m..m] \) for all \( a < b \), then \( \hat{S} \) is multi-transitive.

We can now express the coboundary polynomial of the arrangements. Given indeterminates \( t_1, \ldots, t_N \), we denote \( t = (t_1, \ldots, t_N) \), \( t^n = \prod_{a=1}^N t_a^n \), and \( n! = \prod_{a=1}^N n_a! \). We denote

\[
(5.3) \quad P_S(q, y, t) = \sum_{n \in \mathbb{N}^N} P_{A_S(n)}(q, y) \frac{t^n}{n!},
\]

\[
(5.4) \quad \chi_S(q, t) = \sum_{n \in \mathbb{N}^N} \chi_{A_S(n)}(q) \frac{t^n}{n!},
\]

\[
(5.5) \quad R_S(t) = \sum_{n \in \mathbb{N}^N} r_{A_S(n)} \frac{t^n}{n!},
\]

In the above definition, we adopt the convention \( P_{A_S(0,\ldots,0)}(q, y) = 1 \) (coboundary polynomial of the empty semi-matroid). By (5.1), \( \chi_S(q, t) = P_S(q, 0, t) \) and \( R_S(t) = \chi_S(-1, -t) \), where \( -t = (-t_1, \ldots, -t_N) \).

**Theorem 5.2.** Let \( \hat{S} = (S_{a,b})_{1 \leq a < b \leq N} \) be a \( \binom{N+1}{2} \)-tuple of finite sets of integers, and let \( m = \max(|s|, s \in \cup S_{a,b}) \). Then \( P_{\hat{S}}(q, y, t) \) is related to boxed trees by

\[
(5.6) \quad P_{\hat{S}}(q, y, t) = \left( \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} \sum_{(T,B) \in U(n)} (-1)^{|B|} y^{\text{energy}_{\hat{S}}(T,B)} \right)^{-q}.
\]

In particular,

\[
(5.7) \quad \chi_{\hat{S}}(q, t) = R_{\hat{S}}(-t)^{-q},
\]

and

\[
(5.8) \quad R_{\hat{S}}(t) = \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} \sum_{(T,B) \in U_{\hat{S}}(n)} (-1)^{|n|-|B|}.
\]

Moreover, if \( \hat{S} \) is multi-transitive, then

\[
(5.9) \quad R_{\hat{S}}(t) = \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} |T_{\hat{S}}(n)|.
\]

Note that Equation (5.8) implies Theorem 3.4 (for \( S = S_{1,1} \)) by extracting the coefficient of \( t_1^n t_2^0 \ldots t_N^0 \), and Theorem 4.2 by extracting the coefficient of \( t_1 t_2 \cdots t_N \).

Several examples are treated in Section 6. Let us simply consider here the case \( m = 0 \), which corresponds to graphical arrangements. We denote \( G_{\hat{S}}(n) \) the graph with vertex set \( V(n) \) and edges \( \{u,v\} \) for all \( u = (a,i) \) and \( v = (b,j) \) such that \( S_{a,b} \neq \emptyset \). It follows
easily from the definitions that the number of regions \( r_{A_S}(n) \) is the number of acyclic orientations of \( \hat{G}_S(n) \). Moreover \( \chi_{A_S}(n)(q) \) is the chromatic polynomial of \( \hat{G}_S(n) \), and \( P_{A_S(n)}(q, y) \) is the partition function of the Potts model on \( G = \hat{G}_S(n) \):

\[
P_G(q, y) = \sum_{f:V \to [q]} y^{\text{mono}(f)}
\]

where the sum is over all colorings of the vertices and \( \text{mono}(f) \) is the number of edges of \( G \) with both endpoints of the same color.

Now we consider a few cases of Theorem 5.2. Note that trees in \( U^{(0)}(n) \) are paths with boxes partitioning the vertices. Moreover in each box the vertices are increasing, so \( U^{(0)}(n) \) can be seen as the set of ordered set partitions of \( V(n) \). The \( \hat{S} \)-energy of a subset \( U \) of \( V(n) \) is the number of edges it induces (that is, edges of \( \hat{G}_S(n) \) with both endpoints in \( U \)). For instance, for \( N = 2 \), \( S_{1,1} = S_{22} = \emptyset \) and \( S_{1,2} = \{0\} \), the graph \( G_S(n_1, n_2) \) is the complete bipartite graph \( K_{n_1,n_2} \), and (5.6) gives

\[
\sum_{(n_1,n_2)\in\mathbb{N}^2} P_{K_{n_1,n_2}}(q, y) = \left( \frac{1}{1 + \sum_{(k_1,k_2)\in\mathbb{N}^2\setminus\{(0,0)\}} \frac{y^{k_1k_2}t_1^{k_1}t_2^{k_2}}{k_1!k_2!} } \right)^{-q} = \left( \sum_{(k_1,k_2)\in\mathbb{N}^2} \frac{y^{k_1k_2}t_1^{k_1}t_2^{k_2}}{k_1!k_2!} \right)^{-q}.
\]

Now, for an arbitrary graph \( G = ([N], E) \), we consider \( \hat{S} = (S_{a,b})_{1 \leq a \leq b \leq N} \) with \( S_{a,a} = \{0\} \) for all \( a \in [N] \), and for \( a < b \), \( S_{a,b} = \{0\} \) if \( \{a,b\} \in E \) and \( S_{a,b} = \emptyset \) otherwise. Then, (5.7) gives

\[
(5.10) \quad \chi_G(q) = [t_1t_2 \cdots t_N] R_{\hat{S}}(-t)^{-q}.
\]

This equation relates the proper colorings of \( G \) (left-hand side) and the acyclic orientations of induced subgraphs of \( G \) (right-hand side). This equation imply the interpretation of \( \chi(-q) \) given in [29]. It can be understood through the theory of heaps of pieces (see [32] [19]). Indeed, \( R_{\hat{S}}(t) \) can be seen as the generating function of the heaps of pieces associated to \( G \) (the pieces are the vertices and the pieces overlap if the are adjacent). Hence, by [32] Proposition 5.3, \( R_{\hat{S}}(-t)^{-1} \) is the generating function of trivial heaps, or equivalently, independent sets of \( G \) (that is, sets of non-adjacent vertices).

With this perspective (5.10) simply expresses the fact that a proper \( q \)-coloring of \( G \) is a \( q \)-tuple of independent sets partitioning the vertices. More generally, we get a simple expression for \( \chi_{\hat{S}}(q, t) \):

\[
\chi_{\hat{S}}(q, t) = I(t)^q = \left( \sum_{U \subseteq [N], \text{ independent set of } G \in U} \prod_{i \in U} t_i \right)^q.
\]

\[ \text{The direction of the edge } (u,v) \text{ indicates which side of the hyperplanes } H_{u,v,0} \text{ the region is, or equivalently the inequality between the coordinate } x_u \text{ and } x_v. \]
6. Generating functions

In this section we give equations for the generating functions $P_S(q, y, t)$, $\chi_S(q, t)$, and $R_S(t)$. These equations simply translate the decomposition of boxed trees obtained by deleting the box containing the root.

Let $S = (S_{a,b})_{1 \leq a \leq b \leq N}$ be a $(N+1)$-tuple of finite sets, let $m = \max(|s|, s \in \cup S_{a,b})$.

We first define combinatorial structures encoding admissible cadet sequences for trees in $T^{(m)}(n)$.

**Definition 6.1.** A $(m, N)$-configuration of size $k \in \mathbb{N}^N$, is a pair $\gamma = ((d_1, \ldots, d_{|k|-1}), (u_1, \ldots, u_{|k|}))$ such that $\{u_1, \ldots, u_{|k|}\} = V(k)$, $d_1, \ldots, d_{|k|-1} \in [0..m]$, and if $d_1 = 0$ then $u_i < u_{i+1}$.

We denote $|\gamma| = k$ the size of $\gamma$. The width of $\gamma$ is \text{wid}(\gamma) = d_1 + \cdots + d_{|k|-1} + m + 1$, and the $\hat{S}$-energy of $\gamma$, denoted energy$_\hat{S}(\gamma)$, is the number of pairs $\{i, j\}$ with $1 \leq i < j \leq |k|$ such that $\sum_{p=1}^{i-1} d_p \in (u_i, u_j)$.

**Remark 6.2.** To an admissible cadet sequence $(v_1, \ldots, v_k)$ of a tree $T \in T^{(m)}(n)$, we associate a $(m, N)$-configuration $\gamma = ((d_1, \ldots, d_{k-1}), (u_1, \ldots, u_k))$ defined as follows. Denoting $k = (k_1, \ldots, k_N)$ with $k_a = |\{i \in [m_a] \mid (a, i) \in \{v_1, \ldots, v_k\}\}|$, we set

- $d_i = \text{lsib}(v_{i+1})$ for all $i \in [k-1]$,
- $(u_1, \ldots, u_k)$ is the unique order-preserving relabeling of $(v_1, \ldots, v_k)$ in $V(k)$.

It is clear that $\gamma$ is an $(m, N)$-configuration of size $k$, and that energy$_\hat{S}(\gamma)$ is the $\hat{S}$-energy of the cadet sequence $(v_1, \ldots, v_k)$. Moreover, \text{wid}(\gamma) is the number of children of $v_1, \ldots, v_k$ which are neither in $\{v_1, \ldots, v_k\}$ nor right-siblings of $v_1, \ldots, v_k$.

Let $C^{(m,N)}(k)$ be the set of $(m, N)$-configurations of size $k$, and let $C^{(m,N)} = \bigcup_{k \neq (0, \ldots, 0)} C^{(m,N)}(k)$. Let $C_{\hat{S}}(k)$ be the set of configurations in $C^{(m,N)}(k)$ having $\hat{S}$-energy 0, and let $\hat{C}_{\hat{S}} = \bigcup_{k \neq (0, \ldots, 0)} C_{\hat{S}}(k)$. Finally, let

$$\Gamma_{\hat{S}}(x, y, t) = \sum_{\gamma \in \hat{C}_{\hat{S}}} x^{\text{wid}(\gamma)} y^{\text{energy}_{\hat{S}}(\gamma)} t^{\frac{|\gamma|}{|\gamma|!}},$$

and

$$\Gamma_{\hat{S}}(x, t) = \Gamma_{\hat{S}}(x, 0, t) = \sum_{\gamma \in \hat{C}_{\hat{S}}} x^{\text{wid}(\gamma)} t^{\frac{|\gamma|}{|\gamma|!}}.$$

**Theorem 6.3.** The generating function of coboundary polynomial $P_{\hat{S}}(q, y, t)$ (defined by (5.3)) is equal to $\widetilde{P}_{\hat{S}}(y, t)^{-q}$, where $\widetilde{P}_{\hat{S}}(y, t)$ is the unique series in $\mathbb{Q}[y][[t_1, \ldots, t_N]]$ satisfying

$$(6.1) \quad \widetilde{P}_{\hat{S}}(y, t) = 1 - \Gamma_{\hat{S}}(\widetilde{P}_{\hat{S}}(y, t), y, t).$$

In particular, the generating function of regions $R_{\hat{S}}(t)$ (defined by (5.5)) is the unique series in $\mathbb{Q}[[t_1, \ldots, t_N]]$ satisfying

$$(6.2) \quad R_{\hat{S}}(t) = 1 - \Gamma_{\hat{S}}(R_{\hat{S}}(t), -t).$$

**Example 6.4.** Let $N = 2$ and $S_{1,1} = [-2, 2]$, $S_{1,2} = [-1, 2]$, and $S_{2,2} = \{-2, 0, 1, 2\}$. Then we have $m = 2$ and $C_S = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$, where $\gamma_1 = ((), (v_1))$, $\gamma_2 = ((), (v_2))$.
γ_3 = ((1), (v_2, v_3)), γ_4 = ((2), (v_1, v_2)), and γ_5 = ((2, 1), (v_1, v_2, v_3)) with v_1 = (1, 1), v_2 = (2, 1), v_3 = (2, 2). Thus \( R_S(x, t) = (1 + t_1 + t_2)x^3 + t_2^2 x^4/2 + t_1 t_2 x^3 + t_1 t_2^2 x^6/2 \) and

\[
R_S(t) = 1 + (t_1 + t_2)R_S(t)^3 - t_2^2 R_S(t)^4/2 - t_1 t_2 R_S(t)^5 + t_1 t_2^2 R_S(t)^6/2. 
\]

This gives

\[
R_S(t) = 1 + t_1 + t_2 + 2 t_1^2 + 4 t_1 t_2 + 6 t_2^2 + 12 t_1 + 25 t_1 t_2 + 17/2 t_2^2 + \ldots.
\]

Proof. By Theorem 5.2, \( P_S(q, y, t) = \bar{P}_S(y, t)^{-q} \) for

\[
\bar{P}_S(y, t) := \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \sum_{(T, B) \in U^{(m)}(n)} (-1)^{|B|} y^{\text{energy}_S(T, B)}. 
\]

We now consider the decomposition boxed trees \((T, B) \in U^{(m)}(n)\) for \(|n| > 0\). Consider the cadet sequence \( \beta = (v_1, \ldots, v_k) \in B \) containing the root \( v_1 \) of \( T \). By Remark 6.2, we can associate to \( \beta \) a cadet sequence \( \gamma \). Deleting the vertices \( v_1, \ldots, v_k \) of \( T \) and the right-siblings of \( v_1, \ldots, v_{k-1} \) (which, by definition, are leaves), gives a sequence of \( \text{wid}(\gamma) \) subtrees. Hence, the class \( U^{(m, N)} = \bigcup_{n \in \mathbb{N}} U^{(m)}(n) \) admits the following recursive equation

\[
U^{(m, N)} = 1 + \sum_{\gamma \in C^{(m, N)}} \{\gamma\} \ast \text{Seq}_{\text{wid}(\gamma)}(U^{(m, N)}),
\]

where \( \ast \) denotes the product, and \( \text{Seq}_{\ell} \) denotes the \( \ell \)-sequences construction for labeled combinatorial classes (see e.g. [12, Chapter 2]). This gives

\[
\bar{P}_S(y, t) = 1 + \sum_{\gamma \in C^{(m, N)}} \left(-y^{\text{energy}_S(\gamma)} \frac{t^{|\gamma|}}{|\gamma|!}\right) \times \left(\bar{P}_S(y, t)^{\text{wid}(\gamma)}\right),
\]

which is (6.1). Moreover, (6.1) implies (6.2), because (5.1) gives \( R_S(t) = \bar{P}_S(0, -t) \). \( \square \)

We now explore in more details the case of \( S \)-braid arrangements (case \( N = 1 \) of Theorem 6.3). For a set of integers \( S \), we denote \( C_S = C_{(S)} \). Hence, \( C_S \) is the set of pairs \( \gamma = ((d_1, \ldots, d_{k-1}), (v_1, \ldots, v_k)) \) such that

- \( \{v_1, \ldots, v_k\} = [k] \),
- \( d_1, \ldots, d_{k-1} \in [0..m] \), where \( m = \max\{|s|, s \in S\} \)
- for all \( 0 \leq i < j \leq k \) either \( v_i < v_j \) and \( -\sum_{p=i}^{j-1} d_p \notin S \), or \( v_i > v_j \) and \( \sum_{p=i}^{j-1} d_p \notin S \cup \{0\} \).

Equation (6.2) then gives the following characterization of \( R_S(t) = \sum_{n \geq 0} r_{AS}(n) \frac{t^n}{n!} \):

\[
R_S(t) = 1 - \Gamma_S(R_S(t), -t),
\]

where \( \Gamma_S(x, t) = \sum_{\gamma \in C_S} x^{\text{wid}(\gamma)} \frac{t^{|\gamma|}}{|\gamma|!} \).
**Example 6.5.** For $S = [-3..3] \setminus \{-2, 1\}$, we have $m = 3$ and $C_S = \{\gamma_k, k \geq 1\} \cup \{\gamma'_k\}$, where $\gamma_k = ((2, 0, \ldots, 2), (1, 2, \ldots, k))$ and $\gamma'_k = (1, 2, 1)$. Thus $\Gamma_S(x, t) = \sum_{k \geq 1} t^k x^{2k+2} + t^2 x^5/2 = x^2(e^{tx^2} - 1) + t^2 x^5/2$ and

$$R_S(t) = 1 + R_S(t)^2(1 - e^{-tR_S(t)^2}) - t^2R_S(t)^5/2.$$  

Next, we give an expression for $\Gamma_S(x, t)$ when $S$ is transitive. For $k > 1$ we denote $\mathcal{S}_k$ the set of permutations of $[k]$. For $\pi \in \mathcal{S}_k$, we denote $\text{asc}(\pi) = \{i \in [k-1] | \pi(i) < \pi(i+1)\}$ and $\text{des}(\pi) = \{i \in [k-1] | \pi(i) > \pi(i+1)\}$ the number of ascents and descents of $\pi$. We denote

$$\Lambda(u, v, t) = \sum_{k=1}^{\infty} \sum_{\pi \in \mathcal{S}_k} u^{\text{asc}(\pi)} v^{\text{des}(\pi)} \frac{t^k}{k!}.$$  

This is the generating function of the homogeneous Eulerian polynomials. Clearly, $\Lambda(u, u, t) = \frac{t}{1-tu}$ and $\Lambda(u, 0, t) = e^{tu} - 1$. More generally, it is known [9] that

$$\Lambda(u, v, t) = \frac{e^{tu} - e^{tv}}{ue^{tv} - ve^{tu}}.$$  

**Proposition 6.6.** If $S \subseteq \mathbb{Z}$ is transitive, and $m = \max(|s|, s \in S)$ then

$$\Gamma_S(x, t) = x^{m+1}\Lambda(\mu(x), \nu(x), t),$$  

where $\mu(x) = \sum_{d \in [-m..0] \setminus S} x^d$, and $\nu(x) = \sum_{d \in [m] \setminus S} x^d$. Thus, $R_S(t)$ is the unique solution of

$$R_S(t) = 1 - R_S(t)^{m+1}\Lambda(\mu(R_S(t)), \nu(R_S(t)), -t).$$  

**Example 6.7.** For $S = [-m..m]$, we have $\mu(x) = \nu(x) = 0$. Hence $\Gamma_S(x, t) = tx^{m+1}$, and $R_S(t) = 1 + tR_S(t)^{m+1}$.

**Proof.** Lemma [3.11] gives a simple characterization of $C_S$. Namely $\gamma = ((d_1, \ldots, d_{k-1}), (u_1, \ldots, u_k))$ is in $C_S$ if and only if $u_1, \ldots, u_k$ is a permutation of $[k]$, and for all $i \in [k-1]$, $d_i \in [0..m]$ and either $(v_i < v_{i+1}$ and $d_i \notin S)$ or $(v_i > v_{i+1}$ and $d_i \notin S \cup \{0\})$. Thus, for each ascent $i$ of the permutation $u_1, \ldots, u_k$, $-d_i$ is in $[-m..0] \setminus S$, and for each descent $i$, $d_i$ is in $[m] \setminus S$. This gives (6.4).

Let us first consider the special case $[m] \subseteq S$. Equation (6.7) below is [25, Theorem 9.1] of Postnikov and Stanley.

**Corollary 6.8.** If $[m] \subseteq S \subset [-m..m]$ and $\{s < 0, s \notin S\}$ is closed under addition, then

$$R_S(t) = 1 + R_S(t)^{m+1}\frac{1 - e^{-t\mu(R_S(t))}}{\mu(R_S(t))},$$  

where $\mu(x) = \sum_{d \in [-m..0] \setminus S} x^d$. In particular, if $S = [-\ell..m]$ with $\ell \in [-1..m - 1]$, then

$$R_S(t)^{m-\ell} = \exp\left(\frac{tR_S(t)^{m+1} - R_S(t)^{\ell+1}}{R - 1}\right).$$  

Proof. A set $S$ satisfying the assumptions is transitive. Moreover $\nu(x) = 0$ so that $\Lambda(\mu(x), \nu(x), t) = \frac{\nu(x)}{\mu(x)^{-1}}$. Thus (6.5) gives (6.6). In the particular case $S = [-\ell..m]$ we also have $\mu(x) = \frac{x^{\ell+1}}{x^{-1}}$, and (6.6) readily gives (6.7). \qed

In the special case $S = -S$, we recover [30, Theorem 2.3] of Stanley and [30, Theorem 2.4] (written in a slightly different form) which is credited to Athanasiadis.

**Corollary 6.9 ([30]).** If $S \subseteq \mathbb{Z}$ satisfies $0 \in S$, $\{-s, s \in S\} = S$, and $\mathbb{N} \setminus S$ is closed under addition, then for all $n > 0$,

\begin{equation}
(6.8) \quad r_{AS}(n) = (n - 1)!\sum_{d \in [0..m]\cap S} (x + 1)^d.
\end{equation}

where $m = \max(|s|, s \in S)$. Moreover,

\begin{equation}
(6.9) \quad R_{S \setminus \{0\}}(t) = R_S(1 - e^{-t}).
\end{equation}

Proof. A set $S$ satisfying the assumptions is transitive. Moreover, $\mu(x) = \nu(x)$, so that (6.5) becomes

\begin{equation}
(6.10) \quad R_S(t) = 1 + \frac{t R_S(t)^{m+1}}{1 + t \nu(R_S(t))},
\end{equation}

where $\nu(x) = \sum_{d \in [m]\setminus S} x^d$. This gives $\hat{R}(t) = t \Theta(\hat{R}(t))$, where $\hat{R}(t) = R_S(t) - 1$ and $\Theta(x) = (x + 1)^{m+1} - x \nu(x + 1) = 1 + x \sum_{d \in [0..m] \cap S} (x + 1)^d$. Hence, Lagrange inversion formula gives (6.8). Moreover, $S \setminus \{0\}$ is also transitive, and

\begin{equation}
\Gamma_{S \setminus \{0\}}(x, t) = x^{m+1} \Lambda(\nu(x) + 1, \nu(x), t) = x^{m+1} \frac{e^t - 1}{1 - (e^t - 1)\nu(x)} = \Gamma_S(x, e^t - 1).
\end{equation}

Thus (6.5) becomes

\begin{equation}
R_{S \setminus \{0\}}(t) = 1 + \frac{(1 - e^{-t}) R_{S \setminus \{0\}}(t)^{m+1}}{1 + (1 - e^{-t}) \nu(R_{S \setminus \{0\}}(t))}.
\end{equation}

Comparing this equation with (6.10) gives (6.9). \qed

We now return to the general case $N \geq 1$.

**Theorem 6.10.** Suppose $\tilde{S} = (S_{a,b})_{1 \leq a \leq b \leq N}$ is multi-transitive. Let $\Gamma_1(x, t), \ldots, \Gamma_N(x, t)$ be the series defined by the system of linear equations

\begin{equation}
(6.11) \quad \Gamma_a(x, t) = \Lambda(\mu_{a,0}(x), \nu_{a,0}(x), t) \cdot \left(1 + \sum_{b=1}^{a-1} \nu_{b,0}(x) \Gamma_b(x, t) + \sum_{b=a+1}^{N} \mu_{a,b}(x) \Gamma_b(x, t) \right),
\end{equation}

where for all $1 \leq a \leq b \leq N$, $\mu_{a,b}(x) = \sum_{d \in [-m..0]\setminus S_{a,b}} x^d$, and $\nu_{a,b}(x) = \sum_{d \in [m]\setminus S_{a,b}} x^d$.

Then $\Gamma_{\tilde{S}}(x, t) = x^{m+1} \sum_{a=1}^{N} \Gamma_a(x, t)$ so that

\begin{equation}
R_{\tilde{S}}(t) = 1 - R_{\tilde{S}}(t)^{m+1} \sum_{a=1}^{N} \Gamma_a(R_{\tilde{S}}(t), -t).
\end{equation}
Proof. Let \( C_{S,a} \) be the set of configurations \( \gamma = (d_1, \ldots, d_{k-1}, (v_1, \ldots, v_k)) \in \bar{C}_S \) such that \( v_i \) has the form \((a, i)\) for some \( i \). We claim that \( \Gamma_a(x, t) \) is the generating function of configurations in \( C_{S,a} \). More precisely,

\[
\Gamma_a(x, t) = \sum_{\gamma \in \bar{C}_S} x^{\text{wid}(\gamma)-m-1} t^{|\gamma|} |\gamma|.
\]

Indeed, \( C_{S,a} \) has a simple description (see proof of Theorem 4.6), and (6.11) simply translates the decomposition of configurations in \( C_{S,a} \). at the first \( p \in [k] \) such that \( v_p \) has the form \((b, j)\) with \( b \neq a \) (the term \( 1 \) in (6.11) corresponds to the case where there is no such \( p \)).

Example 6.11. We treat two examples inspired by \([15]\). Suppose first that \( S_{a,a} = \{-1, 0, 1\} \) for all \( a \in [N] \) and \( S_{a,b} = \{-1, 0\} \) for all \( 1 \leq a < b \leq N \). Then (6.11) reads

\[
\Gamma_a(x, t) = t_a(1 + \sum_{b=1}^{a-1} x \Gamma_b(x, t)).
\]

This gives \( \Gamma_a(x, t) = \sum_{k>0} x^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} t_{i_1} \cdots t_{i_k} \), so

\[
\Gamma_{\bar{S}}(x, t) = x \sum_{k>0} x^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} t_{i_1} \cdots t_{i_k} = x \left( \prod_{a=1}^{N} 1 + t_a x \right) - x.
\]

Thus, (6.5) gives

\[
R_{\bar{S}}(t) = \prod_{a=1}^{N} \frac{1}{1 - t_a R_{\bar{S}}(t)}.
\]

Now consider \( \bar{S}' = (S_{a,b}')_{1 \leq a \leq b \leq N} \) with \( S_{a,a}' = \{0\} \) for all \( a \in [N] \) and \( S_{a,b}' = \{0, 1\} \) for all \( 1 \leq a < b \leq N \). Equation (6.11) reads \( \Gamma_a(x, t) = \frac{t_a}{1 - t_a x} (1 + \sum_{b=a+1}^{N} x \Gamma_b(x, t)) \).

hence \( \Gamma_a(x, t) = \sum_{k>0} x^{k-1} \sum_{a=i_1 \leq i_2 < \cdots < i_k \leq N} t_{i_1} \cdots t_{i_k} \), and

\[
\Gamma_{\bar{S}'}(x, t) = x \sum_{k>0} x^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} t_{i_1} \cdots t_{i_k} = x \left( \prod_{a=1}^{N} \frac{1}{1 - t_a x} \right) - x.
\]

Thus, (6.5) gives

\[
R_{\bar{S}'}(t) = \prod_{a=1}^{N} 1 + t_a R_{\bar{S}'}(t).
\]

Note that (6.12) and (6.13) imply that \( R_{\bar{S}}(t) \) and \( R_{\bar{S}'}(t) \) are symmetric functions in \((t_1, \ldots, t_N)\), which is not obvious from the definition. This is a special case of a result proved in \([15]\). In fact, it follows from (5.9) that \( R_{\bar{S}}(t) = 1 + B(1, 1, 0, 1, t) \) and \( R_{\bar{S}'}(t) = 1 + B(1, 1, 0, 1, t) \) for the series \( B(u_1, u_2, v_1, v_2, t) \) considered in \([15]\) which counts trees in \( \bigcup_{p \in \mathbb{N}} T^{(1)}(n) \) according to certain ascent and descent statistics. Accordingly, (6.12) and (6.13) are special cases of the equation given for \( B(u_1, u_2, v_1, v_2, t) \) in \([15]\).
We now state the extensions of Corollary 6.9 to $N > 1$.

**Corollary 6.12.** Suppose that $\tilde{S} = (S_{a,b})_{1 \leq a \leq b \leq N}$ is multi-transitive, and that for all $1 \leq a \leq b \leq N$, the set $S_{a,b}$ contains $0$ and satisfies $\{-s, s \in S_{a,b}\} = S_{a,b}$. Then for all $n \neq (0, \ldots, 0)$,

$$r_{A_{\tilde{S}}}(n) = [t^n] \left( \sum_{a=1}^{N} t_a \right) \left( \prod_{b=1}^{N} g_a(t)^{n_a-1} \right) \det \left( \delta_{i,j} g_i(t) - t_j \frac{\partial g_i(t)}{\partial t_j} \right)_{i,j \in [N]},$$

where $g_a(t) = 1 + \sum_{b=1}^{N} t_b \sum_{d \in [0..m] \cap S_{a,b}} \left( 1 + \sum_{c=1}^{N} t_c \right)^d$, and $\delta_{i,j}$ is the Kronecker delta.

**Example 6.13.** Let $N = 2$ and let $\tilde{S} = (S_{a,b})_{1 \leq a \leq b \leq 2}$ with $S_{1,1} = S_{2,2} = \{-1, 0, 1\}$ and $S_{1,2} = \{0\}$. Then $g_1(t_1, t_2) := 1 + t_1(2 + t_1 + t_2) + t_2$, $g_2(t_1, t_2) := 1 + t_2(2 + t_1 + t_2) + t_1$, the determinant is $(1 + t_1 + t_2)(1 - t_1^2 - t_2^2)$ and Corollary 6.12 gives

$$r_{A_{\tilde{S}}(n_1, n_2)} = [t_1^{n_1} t_2^{n_2}] \left( (1 + t_1 + t_2)(1 - t_1^2 - t_2^2) g_1(t_1, t_2)^{n_1-1} g_2(t_1, t_2)^{n_2-1} \right).$$

**Proof.** Since for all $a \in [N]$, $\Lambda(\nu_{a,a}(x), \nu_{a,a}(x), t_a) = \frac{t_a}{1 - t_a \nu_{a,a}(x)}$, Equation (6.11) can be rewritten as $\Gamma_a(x, t) = t_a(1 + \sum_{b=1}^{N} \nu_{a,b}(x) \Gamma_b(x, t))$. Hence, denoting $R(t) = R_S(t) - 1$ and $R_a(t) = -R_S(t)^{m+1} \Gamma_a(R_S(t), -t)$, we get $R(t) = \sum_{a=1}^{N} R_a(t)$ and $R_a(t) = t_a g_a(R_1, \ldots, R_N)$, where

$$g_a(r_1, \ldots, r_N) = \left( 1 + \sum_{c=1}^{N} r_c \right)^{m+1} - \sum_{b=1}^{N} r_b \nu_{a,b} \left( 1 + \sum_{c=1}^{N} r_c \right) = 1 + \sum_{b=1}^{N} r_b \sum_{d \in [0..m] \cap S_{a,b}} \left( 1 + \sum_{c=1}^{N} r_c \right)^d.$$

Applying Lagrange inversion formula [14] [16] gives the result. $\square$

**Corollary 6.14.** Suppose that $\tilde{S} = (S_{a,b})_{1 \leq a \leq b \leq N}$ is multi-transitive, and that for some $a \in [N]$, $S_{a,a}$ contains $0$ and satisfies $\{-s, s \in S_{a,a}\} = S_{a,a}$. Let $\tilde{S'}$ be the same tuple as $\tilde{S}$ except $S_{a,a}$ is replaced by $S_{a,a} \setminus \{0\}$. Then,

$$R_{\tilde{S}'}(t) = R_S(t_1, \ldots, t_{a-1}, 1 - e^{-t_a}, t_{a+1}, \ldots, t_N).$$

**Proof.** Let us denote $\Gamma_1(x, t), \ldots, \Gamma_N(x, t)$ the series satisfying (6.11) for the tuple $\tilde{S}$ and $\Gamma_1'(x, t), \ldots, \Gamma_N'(x, t)$ their analogues for $\tilde{S}'$. As in the proof of Corollary 6.9 we get for all $i \in [N]$ $\Gamma_i'(x, t) = \Gamma_i'(x, t')$, where $t' = (t_1, \ldots, t_{a-1}, e^{t_a} - 1, t_{a+1}, \ldots, t_N)$. Thus $\Gamma_{\tilde{S}}(x, t) = \Gamma_{\tilde{S}'}(x, t')$, and $\Gamma_{\tilde{S}'}(x, -t') = \Gamma_{\tilde{S}}(x, -t'')$, for $t'' = (t_1, \ldots, t_{a-1}, 1 - e^{-t_a}, t_{a+1}, \ldots, t_N)$. Hence (6.2) implies $R_{\tilde{S}'}(t) = R_{\tilde{S}}(t'')$. $\square$

**Corollary 6.15.** Suppose that $\tilde{S} = (S_{a,b})_{1 \leq a \leq b \leq N}$ is multi-transitive, and that for all $a < b$, $S_{a,b} = S$ for some set $S$ containing $0$ and such that $\{-s, s \in S\} = S$. Then

$$\Gamma_{\tilde{S}}(x, t) = x^{m+1} \frac{\Delta(x, t)}{1 - \nu(x) \Delta(x, t)},$$

where $\Delta(x, t) = \text{det} \left( \delta_{i,j} g_i(t) - t_j \frac{\partial g_i(t)}{\partial t_j} \right)_{i,j \in [N]}.$
where \( \Delta(x, t) = \sum_{a=1}^{N} \frac{\Lambda(\mu_a(x), \nu_a(x), t_a)}{1 + \nu(x) \Lambda(\mu_a(x), \nu_a(x), t_a)}, \mu_a(x) = \sum_{d \in [-m..0]\setminus S_{a,a}} x^d, \nu_a(x) = \sum_{d \in [m]\setminus S_{a,a}} x^d, \) and \( \nu(x) = \sum_{d \in [m]\setminus S} x^d. \)

**Example 6.16.** Let \( N = 2 \) and \( \hat{S} = (S_{a,b})_{1 \leq a \leq b \leq 2} \) with \( S_{1,1} = \{-1, 0, 1\}, S_{2,2} = \{0\} \) and \( S_{1,2} = \{0\} \). Then with the notation of Corollary [6.15], we have \( \nu(x) = x, \Delta(x, t) = \frac{t_1}{1 + t_1 x} + \frac{e^{t_2 x} - 1}{x + (e^{t_2 x} - 1)x} \) and

\[
\Gamma_{\hat{S}}(x, t) = -\frac{x (2t_1 x e^{t_2 x} - t_1 x + e^{t_2 x} - 1)}{t_1 x e^{t_2 x} - t_1 x - 1}.
\]

**Proof.** Equation (6.11) can be rewritten as

\[
\Gamma_a(x, t) = \Lambda(\mu_a(x), \nu_a(x), t_a)(1 + \nu(x) (\Gamma_{\hat{S}}(x, t)/x^{m+1} - \Gamma_a(x, t))).
\]

Thus \( \Gamma_a(x, t) = \frac{\Lambda(\mu_a(x), \nu_a(x), t_a)}{1 + \nu(x) \Lambda(\mu_a(x), \nu_a(x), t_a)} (1 + \nu(x) \Gamma_{\hat{S}}(x, t)/x^{m+1}), \) and

\[
\Gamma_{\hat{S}}(x, t) = x^{m+1} \sum_{a=1}^{N} \Gamma_a(x, t) = \Delta(x, t) (x^{m+1} + \nu(x) \Gamma_{\hat{S}}(x, t)).
\]

\( \square \)

**Remark 6.17.** Our results for the number of regions of a multi-transitive arrangement \( \mathcal{A}_{\hat{S}}(n) \) have been derived from (5.8) using the decomposition of box trees in \( \mathcal{U}_{\hat{S}}(n) \). They could alternately be obtained from (5.9) using the decomposition of trees in \( \mathcal{T}_{\hat{S}}(n) \). When \( S_{a,b} = -S_{a,b} \) for all \( a, b \in [N] \) one can simply use the decomposition obtained by deleting the root. In the general case, the decomposition would correspond to deleting all the vertices in the longest cadet sequence starting at the root.

7. Proofs

As explained above, Theorem 5.2 implies Theorems 3.4 and Theorems 4.2. It remains to prove Theorem 5.2. The proof breaks into three steps corresponding to Lemmas 7.1, 7.3, and 7.5 below.

We first express the coboundary polynomial of any deformation of the braid arrangement as a weighted count of graphs. We denote \( \mathcal{G}_n \) the set of graphs (without loops nor multiple edges) with vertex set \( [n] \).

**Lemma 7.1.** Let \( n \in \mathbb{N} \), and let \( S = (S_{u,v})_{1 \leq u < v \leq n} \) be a \( \binom{n}{2} \)-tuple of finite sets of integers, and let \( m = \max(|S|, s \in \cup S_{u,v}) \). The coboundary polynomial of \( \mathcal{A}_{S} \) is

\[
P_{\mathcal{A}_{S}}(q, y) = \sum_{G \in \mathcal{G}_n} (y - 1)^{e(G)} q^{c(G)} |W_{S}(G)|,
\]

where \( e(G) \) and \( c(G) \) are the number of edges and connected components of \( G \) respectively, and \( W_{S}(G) \) is the set of tuples \( (x_1, \ldots, x_n) \in \mathbb{Z}^n \) such that

- for all edge \( \{u, v\} \) of \( G \) with \( u < v \), \( x_u - x_v \) is in \( S_{u,v}, \)
• for all vertex \( v \) of \( G \) such that \( v \) is smallest in its connected component, \( x_v = 0 \).

**Example 7.2.** Let \( n = 3 \) and \( S_{1,2} = S_{1,3} = \{-2, 1\} \) and \( S_{2,3} = \{-2, -1, 1\} \). Let \( G \) be the graph with vertex set \([3]\) and edges \( \{1, 2\} \) and \( \{2, 3\} \). Then \( \mathcal{W}_S(G) = \{(0, -1, -2), (0, -1, 0), (0, -1, 1), (0, 2, 1), (0, 2, 3), (0, 2, 4)\} \).

**Proof.** To a subarrangement \( B \subseteq A_S \), we associate the graph \( G_B \in \mathcal{G}_n \) with arcs \( \{u, v\} \) for all \( u < v \) such that there exists \( s \in S_{u,v} \) such that \( H_{u,b,s} \in B \). We say that \( B \) is **central** if \( \bigcap_{H \in B} H \neq \emptyset \). If \( B \) is central, for each edge \( \{u, v\} \) of \( G_B \), with \( u < v \), there is a unique value \( s \in S_{u,v} \) such that \( H_{u,v,s} \in B \), and we denote this value \( B(u,v) \). We also denote \( B(v,u) = -B(u,v) \). Clearly, a point \((x_1, \ldots, x_n)\) is in \( \bigcap_{H \in B} H \) if and only if for any path \( v_1, v_2, \ldots, v_k \) in \( G_B \),

\[
x_{v_1} - x_{v_k} = \sum_{i=1}^{k-1} B(v_i, v_{i+1}).
\]

Hence, there is a unique point \( x_B = (x_1, \ldots, x_n) \) in \( \bigcap_{H \in B} H \) such that \( x_v = 0 \) for all \( v \in [n] \) such that \( v \) is the smallest vertex in its connected component of \( G_B \). Moreover, \( \dim(\bigcap_{H \in B} H) = c(G_B) \). Note also that \( x_B \) is in \( \mathcal{W}_S(G_B) \), and that \( B \) is uniquely determined by the pair \((G_B, x_B)\). Lastly, any pair \((G, x)\) where \( G \in \mathcal{G}_n \) and \( x \in \mathcal{W}_S(G) \) comes from a central subarrangement \( B \). Thus,

\[
P_{A_S}(q, y) = \sum_{B \subseteq A_S \text{ central}} (y - 1)^{|B|} q^{\dim(\bigcap_{H \in B} H)}
\]

\[= \sum_{(G, x) \in \mathcal{G}_n, x \in \mathcal{W}_S(G)} (y - 1)^{|G|} q^{c(G)}
\]

\[= \sum_{G \in \mathcal{G}_n} (y - 1)^{|G|} q^{c(G)} |\mathcal{W}_S(G)|.
\]

\[\square\]

Our second step relates the generating function \( P_S(q, y, t) \) of coboundary polynomials, to a generating function \( Z_{\hat{S}}(\delta, y, t) \) of tuples of integers. We fix \( N > 0 \) and a \( (N+1) \)-tuple \( \hat{S} = (S_{a,b})_{1 \leq a < b \leq N} \) of finite sets of integers. As before, \( m = \max(|s|, \ s \in \bigcup S_{a,b}) \) and for \( \mathbf{n} \in \mathbb{N}^N \) and \( u, v \in V(\mathbf{n}) \) with \( u = (a, i) \), \( v = (b, j) \) and \( u < v \), we denote \( S_{u,v} = S_{a,b} S_{v,u} = S_{a,b} S_{u,v} = \{s \geq 0 \mid -s \in S_{a,b}\} \), and \( S_{v,u} = \{s > 0 \mid s \in S_{a,b} \} \cup \{0\} \).

For a positive integer \( \delta \), and \( \mathbf{n} \in \mathbb{N}^N \), we denote

\[Z_{\hat{S}, \mathbf{n}}(\delta, y) = \sum_{\mathbf{x} = (x_v) \in V(\mathbf{n}) \in [\delta]^{[n]}} y^{\text{energy}_{\hat{S}}(\mathbf{x})},
\]

where \( \text{energy}_{\hat{S}}(\mathbf{x}) \) is number of pairs \((u, v) \in V(\mathbf{n})^2 \), with \( u < v \), such that \( x_u - x_v \in S_{u,v} \). For instance, the \( \hat{S} \)-energy of the tuple \( \mathbf{x} \) in Figure 7 is 1. By convention, we set \( Z_{\hat{S}, (0, \ldots, 0)}(\delta, y) = 1 \).

**Lemma 7.3.** The generating functions \( P_{\hat{S}}(q, y, t) \) and

\[Z_{\hat{S}}(\delta, y, t) = \sum_{\mathbf{n} \in \mathbb{N}^N} Z_{\hat{S}, \mathbf{n}}(\delta, y) \frac{t^n}{n!},
\]
Recall that the arrangement

\[ \text{(7.1)} \]

is related by

\[ \frac{1}{q} \log(P_\mathcal{S}(q, y, t)) = \lim_{\delta \to \infty} \frac{1}{\delta} \log(Z_\mathcal{S}(\delta, y, t)). \]

Equation (7.1) is to be understood as an identity for formal power series in \( t_1, \ldots, t_N \):

- the limit is taken coefficient by coefficient in \( t_1, \ldots, t_N \),
- for a series in formal power series \( A(t_1, \ldots, t_N) \) such that \( A(0, \ldots, 0) = 1 \), we denote \( \log(A(t)) \) the formal power series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(A(t)-1)^n}{n} \).

**Proof.** Let \( \mathcal{G}_n \) be the set of graphs with vertex set \( V(n) \), and let \( \mathcal{G}^{(N)} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n \).

For \( G \in \mathcal{G}_n \), we denote \( \mathcal{W}_\mathcal{S}(G) \) the set of tuples \( (x_v)_{v \in V(n)} \in \mathbb{Z}^{|n|} \) such that

- for all edge \( \{u, v\} \) of \( G \) with \( u < v \), \( x_u - x_v \in S_{u,v} \),
- for all vertex \( v \) of \( G \) which is smallest in its connected component, \( x_v = 0 \).

Recall that the arrangement \( \mathcal{A}_\mathcal{S}(n) \) identifies with the arrangement \( \mathcal{A}_\mathcal{S}(n) \), where the tuple \( \mathcal{S}(n) \) is given by (5.2). Up to this identification, Lemma 7.1 gives

\[ P_{\mathcal{A}_\mathcal{S}(n)}(q, y) = \sum_{G \in \mathcal{G}_n} (y - 1)^{e(G)} q^{c(G)} |\mathcal{W}_\mathcal{S}(G)|, \]

hence

\[ \text{(7.2)} \]

\[ P_{\mathcal{S}}(q, y, t) = \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} \sum_{G \in \mathcal{G}_n} (y - 1)^{e(G)} q^{c(G)} |\mathcal{W}_\mathcal{S}(G)|. \]

We now apply the multivariate exponential formula. We think of \( \mathcal{G}^{(N)} \) as the combinatorial class of graphs with \( N \) types of vertices, with the vertices of each type being well-labeled (that is, the \( n_a \) vertices of type \( a \) have distinct labels in \( [n_a] \)). The size of \( G \in \mathcal{G}_n \) is \( n = (n_1, \ldots, n_N) \) and the weight of \( G \) is \( (y - 1)^{e(G)} q^{c(G)} |\mathcal{W}_\mathcal{S}(G)| \). The weight is multiplicative over connected components (and unchanged by order preserving relabeling of the vertices of each type). Hence the multivariate exponential formula applies (see e.g. [31]), and taking the logarithm of \( P_{\mathcal{S}}(q, y, t) \) amounts to selecting the connected graphs in \( \mathcal{G}^{(N)} \). This gives,

\[ \text{(7.3)} \]

\[ \frac{1}{q} \log(P_{\mathcal{S}}(q, y, t)) = \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} \sum_{G \in \mathcal{G}_n} \text{ connected} (y - 1)^{e(G)} |\mathcal{W}_\mathcal{S}(G)|. \]
Next, observe that
\[
|Z_{S,n}(\delta,y)| = \sum_{(x_v)_{v \in V(n)} \in [\delta][n]} \prod_{u,v \in V(n), u < v} \left(1 + (y - 1) \cdot 1_{x_u - x_v \in S_{u,v}}\right),
\]
\[
= \sum_{(x_v)_{v \in V(n)} \in [\delta][n]} \sum_{G \in G_n} (y - 1)^{e(G)} \left(\prod_{\{u,v\} \text{ edge of } G, u < v} 1_{x_u - x_v \in S_{u,v}}\right),
\]
\[
= \sum_{G \in G_n} (y - 1)^{e(G)} |W_{S,\delta}(G)|,
\]
where 1 is the indicator function, and \(W_{S,\delta}(G)\) is the set of tuples \((x_v)_{v \in V(n)} \in [\delta][n]\) such that for all edge \(\{u,v\}\) of \(G\) with \(u < v\), \(x_u - x_v \in S_{u,v}\). The graph weight \((y - 1)^{e(G)} |W_{S,\delta}(G)|\) is multiplicative over connected components, hence by the multivariate exponential formula,
\[
\log(Z_{S,\delta}(t)) = \log \left(\sum_{n \in \mathbb{N}^V} \frac{t^n}{n!} \sum_{G \in G_n} (y - 1)^{e(G)} |W_{S,\delta}(G)|\right),
\]
\[
= \sum_{n \in \mathbb{N}^V} \frac{t^n}{n!} \sum_{G \in G_n, \text{connected}} (y - 1)^{e(G)} |W_{S,\delta}(G)|.
\]
(7.4)

It only remains to prove that for any connected graph \(G \in G_n\),
\[
\lim_{\delta \to \infty} \frac{1}{\delta} |W_{S,\delta}(G)| = |W_{S}(G)|.
\]
(7.5)

It is easy to see that, the tuples in \(W_{S,\delta}(G)\) are translations of tuples in \(W_{S}(G)\), and the number of translations is of order \(\delta\). More precisely,
\[
W_{S,\delta}(G) = \{(x_v + \theta)_{v \in V(n)} \mid (x_v)_{v \in V(n)} \in W_{S}(G), \text{ and } 1 - \min(x_v)_{v \in V(n)} \leq \theta \leq \delta - \max(x_v)_{v \in V(n)}\}.
\]
The tuples above are all distinct, and for any \((x_v)_{v \in V(n)} \in W_{S}(G)\), \(\max(x_v)_{v \in V(n)} - \min(x_v)_{v \in V(n)} \leq m \cdot |n|\). Thus
\[
(\delta - m \cdot |n|) |W_{S}(G)| \leq |W_{S,\delta}(G)| \leq \delta |W_{S}(G)|.
\]
This shows (7.5), and completes the proof. \(\square\)

**Remark 7.4.** The proof of Lemma 7.3 is reminiscent of Mayers’ theory of cluster integrals (see e.g. [8, 20, 21]). In this perspective, the right-hand side of (7.1) corresponds to the pressure of the infinite volume limit of a discrete gas model (where particles of type \(a\) and \(b\) interacts according to a soft-core potential of shape \(S_{a,b}\), and energy of interaction \(y\)). Alexander Postnikov also pointed out to the author that Lemma 7.3 could alternatively be obtained by using the finite field method pioneered by Athanasiadis [4] and adapted to the calculation of coboundary polynomials in [1]. However, the situation of deformed braid arrangement is distinguished by the fact that the parameter \(q\) appears merely as an exponent of the generating function \(P_S(q,y,t)\); see [5, 7]. This fact, which is a direct consequence of Lemma 7.2 and already appears in [30, Theorem 1.2], allows one to focus the remaining analysis on a single value of \(q\), namely \(+\infty\).
Our last step, relates the generating function $Z_S(\delta, y, t)$ of point configurations to the generating function of boxed trees.

**Lemma 7.5.** Let

$$U_S(y, t) = \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} \sum_{(T,B) \in U^{(m)}(n)} (-1)^{|B|} y^\text{energy}_S(T,B),$$

and

$$U_S^*(y, t) = \sum_{n \in \mathbb{N}^N} \frac{t^n}{n!} \sum_{(T,B) \in U^{(m)}(n)} (|B| + \text{leaf}(T))(\alpha + 1) |\text{energy}_S(T,B)|,$$

where $\text{leaf}(T)$ is the number leaves of $T$. For all $n \in \mathbb{N}^N$, and for all $\delta > m \cdot |n|$, (7.6)

$$Z_{S,n}(\delta, y) = [t^n]U_S(y, t)^{-\delta - m - 2}U_S^*(y, t).$$

We will use the following fact about tuples of rooted plane trees.

**Claim 7.6.** Let $\alpha, w_1, \ldots, w_r$ be positive integers. Let $\tau(\alpha; w_1, \ldots, w_r)$ be the set of tuples $(T_1, \ldots, T_\alpha)$, where $T_1, \ldots, T_\alpha$ are rooted plane trees, and $T_\alpha$ has a marked vertex, such that, denoting $c_i, \ldots, c_{i,k_i}$ the number of children of the nodes of $T_i$ in prefix order, one has

$$(c_1, 1, \ldots, c_{1,k_1}, c_2, 1, \ldots, c_{2,k_2}, \ldots, c_\alpha, 1, \ldots, c_{\alpha,k_\alpha}) = (w_1, \ldots, w_r).$$

Then,

$$|\tau(\alpha; w_1, \ldots, w_r)| = \binom{\alpha + w_1 + \cdots + w_r}{r}.$$  

**Proof.** The proof is represented in Figure 8. Let $P$ be the set of lattice paths on $\mathbb{Z}$ starting at 0, and having every step greater or equal to 0. Let $P_{-1}$ be the set of paths in $P$ ending at 0, and let $P_{+1} \subset P_{-1}$ be the subset of paths remaining non-negative until the last step. Recall that the map $\phi$ which associates to a rooted plane tree $T$ the path $P$ with steps $c_1 - 1, c_2 - 1, \ldots, c_n - 1$ where $c_1, \ldots, c_n$ are the number of children of the vertices of $T$ taken in prefix order is a bijection between rooted plane trees and $P_{-1}$ (see e.g. [11] Chapter 5.3]). Moreover by the cycle lemma, there is a $n$-to-1 correspondence between the paths with $n$ steps in $P_{-1}$ and the paths with $n$ steps in $P_{+1}$. Thus the map $\phi$ induces a bijection between $P_{-1}$ and rooted plane trees with a marked vertex.

Now let $P(\alpha; w_1, \ldots, w_r)$ be the set of path in $P$ having $\alpha + w_1 + \cdots + w_r - r$ steps 0, and $r$ non-negative steps $w_1 - 1, \ldots, w_r - 1$ in this order. Clearly, $|P(\alpha; w_1, \ldots, w_r)| = (\alpha + w_1 + \cdots + w_r)$, and paths in $P(\alpha; w_1, \ldots, w_r)$ ends at $-\alpha$. We consider the decomposition of paths in $P(\alpha; w_1, \ldots, w_r)$ at the first time they reach $-1, -2, \ldots, -\alpha + 1$, as represented in Figure 8. This gives a bijection between $P(\alpha; w_1, \ldots, w_r)$ and the set of tuples $(P_1, \ldots, P_\alpha)$ such that $P_1, \ldots, P_{\alpha-1} \in P_{+1}, P_\alpha \in P_{-1}$ and there is a total of $\alpha + w_1 + \cdots + w_r - r$ steps 0, and $r$ non-negative steps $w_1, \ldots, w_r$ in this order. Combining this decomposition with $\phi$ gives a bijection between $P(\alpha; w_1, \ldots, w_r)$ and $\tau(\alpha; w_1, \ldots, w_r)$, thereby proving the claim. \qed
Lastly, there are \( \sum \) ways to choose the set partition \((L_1, \ldots, L_n)\). Hence

\[
U_S(y, t) = \sum_{R \in \mathcal{R}} \prod_{v \text{ node of } R} \left( \frac{1}{|\gamma|!} \sum_{\gamma \in \mathcal{C}(m,N) \mid \text{wid}(\gamma) = \text{child}(v)} y^{\text{energy}_S(\gamma)} t^{\frac{|\gamma|}{|\gamma|!}} \right),
\]

where \(\mathcal{R}\) is the set of rooted plane trees and \(\mathcal{C}(m,N)\) is the set of \((m, M)\)-configurations. Similarly,

\[
U_S(y, t) = \sum_{R \in \mathcal{R}} v(R) \prod_{v \text{ node of } R} \left( \frac{1}{|\gamma|!} \sum_{\gamma \in \mathcal{C}(m,N) \mid \text{wid}(\gamma) = \text{child}(v)} y^{\text{energy}_S(\gamma)} t^{\frac{|\gamma|}{|\gamma|!}} \right),
\]

where \(v(R)\) is the number of vertices of \(R\).

Next, we express \(Z_{S,n}(\delta, y)\) in terms of \((m, N)\)-configurations. Let \(n \in \mathbb{N}^N\), and let \(x = (x_v)_{v \in V(n)} \in [\delta]^{|n|}\). Intuitively, we think of each coordinate \(x_v\) as the position of a particle in the space \([\delta]\), and we will distinguish \textit{runs} which are groups of particles.
that are close to one another. This is represented in Figure 7. Let \( v'_1, \ldots, v'_{|n|} \in V(n) \) be defined by \( \{v'_1, \ldots, v'_{|n|}\} = V(n) \), and the conditions \( x_{v'_i} \leq x_{v'_{i+1}} \), and if \( x_{v'_i} = x_{v'_{i+1}} \), then \( v'_i < v'_{i+1} \). We denote \( d_i = x_{v'_{i+1}} - x_{v'_i} \) for all \( i \in [|n| - 1] \), and adopt the convention \( d_0 = d_{|n|} = \infty \). A run of \( x \) is a subsequence \( \rho = (v'_i, v'_{i+1}, \ldots, v'_j) \), with \( 1 \leq i < j \leq |n| \), such that \( d_{i-1} > m, d_j > m \), and for all \( i \leq p < j, d_p \leq m \). We define the position of \( \rho \) as \( \text{pos}(\rho) = x_{v'_i} \), the width of \( \rho \) as \( \text{wid}(\rho) = x_{v'_j} - x_{v'_i} + m + 1 \), the labels of \( \rho \) as \( \text{lab}(\rho) = \{v'_i, \ldots, v'_j\} \), and the size of \( \rho \) as \( |\rho| = k = (k_1, \ldots, k_N) \) where \( k_a = |\{(a, i) \in \text{lab}(\rho)\}| \). For instance, the tuple \( x \) represented in Figure 7 has four runs having position 3, 7, 12, and 19 respectively, width 4, 3, 6, and 5 respectively, and size \((2, 0), (0, 1), (3, 2), \) and \((1, 1) \) respectively. Lastly, the configuration of \( \rho \) is \( \text{config}(\rho) = ((d_1, d_{i+1}, \ldots, d_{j-1}), (u_1, \ldots, u_j)) \), where \((u_1, \ldots, u_j)\) is the unique order preserving relabeling of \((v'_1, \ldots, v'_j)\) in \( V(k) \). For instance, in Figure 7, \( \text{config}(\rho_3) = ((1, 0, 2, 0), ((2, 2), (1, 1), (1, 2), (1, 3), (2, 1)) \). Note that \( \gamma = \text{config}(\rho) \) is in \( C^{1(n)}(k) \), and \( \text{wid}(\rho) = \text{wid}(\gamma) \). Moreover it is easy to see that

\[
\text{energy}_S(x) = \sum_{i=1}^r \text{energy}_S(\text{config}(\rho_i)),
\]

where \( \rho_1, \ldots, \rho_r \) are the runs of \( x \) (because pairs of particles at distance greater than \( m \) do not contribute to the \( S \)-energy). Moreover, the tuple \( x \) is completely determined by the positions, labels, and configurations of its runs \( \rho_1, \ldots, \rho_r \). The configurations of the runs are arbitrary, and given the configurations \( \gamma_1, \ldots, \gamma_r \) of the runs there are

\[
\binom{\delta + m + r - \text{wid}(\gamma_1) - \cdots - \text{wid}(\gamma_r)}{r}
\]

ways to choose the positions \((p_1, \ldots, p_r) \in [\delta]^r \) (since the only constraints are \( p_i + \text{wid}(\gamma_i) \leq p_{i+1} \) for all \( i \in [r - 1] \), and \( p_r + \text{wid}(\gamma_r) \leq \delta + m + 1 \)). Also, there are

\[
\prod_{i=1}^r \frac{n!}{|\gamma_i|!}
\]

ways to choose the labels. Thus,

\[
Z_{S,n}(\delta, y) = n! \sum_{r=0}^\infty \sum_{\gamma_1, \ldots, \gamma_r \in C^{1(n)}} \binom{\delta + m + r - \text{wid}(\gamma_1) - \cdots - \text{wid}(\gamma_r)}{r} \frac{y^{\text{energy}_S(\gamma_i)}}{|\gamma_i|!}.
\]

In order to prove (7.6), we will now consider negative values of \( \delta \). Let us denote \( \text{Pol}_r(x) = \frac{x(x-1) \cdots (x-r+1)}{r!} \). This is a polynomial in \( x \), such that for all \( x \in \mathbb{N} \), \( \binom{x}{r} = \text{Pol}_r(x) \). Let

\[
Z_{S,n}(\delta, y) = n! \sum_{r=0}^\infty \sum_{\gamma_1, \ldots, \gamma_r \in C^{1(n)}} \text{Pol}_r(\delta + m + r - \text{wid}(\gamma_1) - \cdots - \text{wid}(\gamma_r)) \prod_{i=1}^r \frac{y^{\text{energy}_S(\gamma_i)}}{|\gamma_i|!}.
\]

This is a polynomial in \( \delta \) and \( y \) which coincides with \( Z_{S,n}(\delta, y) \) for all integer \( \delta > m \cdot |n| \), because any \( \gamma \in C^{1(n)} \) satisfies \( \text{wid}(\gamma) - 1 \leq m \cdot |\gamma| \). It remains to prove that for all \( \delta \),

\[
Z_{S,n}(\delta, y) = |t^n| U_S(y, t)^{-\delta - m - 2} U^*_S(y, t).
\]
We observe that both sides of (7.9) are polynomials in $\delta$. Indeed, $\hat{U}_S(y, (0, \ldots, 0)) = 1$, hence the series

$$U_\delta(y, t) = \exp(-\delta \log(U_S(y, t))) = \exp\left(\delta \sum_{k \geq 1} \frac{(1 - U_S(t))^k}{k}\right)$$

has coefficients which are polynomial in $\delta$. Thus, in order to prove (7.9), it suffices to prove it for infinitely many values of $\delta \in \mathbb{C}$. Let $\alpha$ be a positive integer, let $\delta = -m - 1 - \alpha$, and let

$$\tilde{Z}_S(\delta, y, t) = \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \tilde{Z}_{S,n}(\delta, y).$$

We have

$$\tilde{Z}_S(\delta, y, t) = \sum_{r=0}^{\infty} \sum_{\gamma_1, \ldots, \gamma_r \in \mathbb{C}(m, N)} \text{Pol}_r \left(\alpha + \widetilde{\text{wid}}(\gamma_1) + \cdots + \widetilde{\text{wid}}(\gamma_r)\right) \prod_{i=1}^{r} \frac{y_{\text{energy}}(\gamma_i)_{\frac{1}{\gamma_i}}}{|\gamma_i|!}.$$

Using Claim 7.6 gives

$$\tilde{Z}_S(\delta, y, t) = \sum_{r=0}^{\infty} (-1)^r \left(\sum_{\gamma_1, \ldots, \gamma_r \in \mathbb{C}(m, N)} \tau(\alpha; \widetilde{\text{wid}}(\gamma_1), \ldots, \widetilde{\text{wid}}(\gamma_r)) \prod_{i=1}^{r} \frac{y_{\text{energy}}(\gamma_i)_{\frac{1}{\gamma_i}}}{|\gamma_i|!}\right)^{\alpha - 1} \times \left(\sum_{R \in \mathbb{R}} \prod_{v \text{ node of } R} \sum_{\gamma \in \mathbb{C}(m, N), \widetilde{\text{wid}}(\gamma) = \text{child}(v)} \frac{y_{\text{energy}}(\gamma)_{\frac{1}{\gamma}}}{|\gamma|!}\right).$$

for all $\delta \leq -m - 2$. Thus, $\tilde{Z}_S(\delta, y, t) = U_\delta(y, t)^{-\delta - m - 2} U_{\delta}^\bullet(y, t)$ for all $\delta \in \mathbb{C}$, and extracting the coefficient of $t^n$ gives (7.9). 

We can now complete the proof of Theorem 5.2: By Lemma 7.3,

$$\lim_{\delta \to \infty} \frac{Z_{S,\delta}(\delta, y, t)}{U_\delta(y, t)^{\delta - m - 2} U_{\delta}^\bullet(y, t)} = 1,$$
Hence, the number of \( \alpha \) and \( \beta \) parenthesis systems. An example is represented in Figure 9. A first recall a classical encoding of the regions of \( T \) the general case in Subsection 8.3. cases of the Shi, semiorder and Linial arrangements in Subsection 8.2, before treating with a convenient bijection between parenthesis systems and trees. Then, we treat the permutation \( \pi \) from a parenthesis system \( w \) the regions of \( T \) is represented in Figures 10 and 13. we say that the letter \( \alpha \) or \( \beta \) \((m+1)n!/(mn+1)!\) \( m \)-parenthesis systems of size \( n \), and that such words bijectively encode rooted plane \((m+1)\)-ary trees with \( n \) nodes (see e.g. [31, Chapter 5.3]). A \( m \)-sketch of size \( n \) is a word \( \tilde{w} \) obtained from a parenthesis system \( w \) by replacing the \( i \)th letter \( \alpha \) by the letter \( \alpha_{\pi(i)} \) for some permutation \( \pi \) of \([n]\). We denote \( \mathcal{D}(m)(n) \) the set of \( m \)-sketches of size \( n \). Clearly, \( |\mathcal{D}(m)(n)| = n!\text{Cat}(m)(n) = \frac{(m+1)n!}{m(mn+1)!} = |\mathcal{T}(m)(n)| \). We now describe bijections between the regions of \( A_{[-m..m]}(n) \), and the sets \( \mathcal{D}(m)(n) \) and \( \mathcal{T}(m)(n) \). The case \( m = 1, n = 3 \) is represented in Figures 10 and 13. We first need to annotate our sketches. Let \( A(m)(n) \) be the alphabet made of the \((m+1)n\) letters \( \{\alpha_i^{(s)} \mid i \in [n], s \in [0..m]\} \). We call \( \alpha \)-letters the letters \( \alpha_i^{(0)} \) for \( i \in [n] \), and \( \beta \)-letters the letters \( \alpha_i^{(s)} \) for \( i \in [n], s \in [m] \). For a word \( \tilde{w} \) on the alphabet \( A(m)(n) \), we say that the letter \( \alpha_i^{(s)} \) is active in \( \tilde{w} \) if \( s < m \), and \( \alpha_i^{(s)} \) appears in \( \tilde{w} \) but \( \alpha_i^{(s+1)} \) does
not. The annotation of a sketch \( \hat{w} = \hat{w}_1 \cdots \hat{w}_{(m+1)n} \) is the word \( \hat{w} = \hat{w}_1 \cdots \hat{w}_{(m+1)n} \) obtained by applying the following rule for \( p = 1, 2, \ldots, (m + 1)n \): if \( \hat{w}_p = \alpha_i \) then set \( \hat{w}_p = \alpha_i^{(0)} \), while if \( \hat{w}_p = \beta \) then set \( \hat{w}_p = \alpha_i^{(s+1)} \), where \( \alpha_i^{(s)} \) is the first active letter in \( \hat{w}_1 \cdots \hat{w}_{p-1} \) (it is easy to see that there is always such a letter). We denote \( D^{(m)}(n) \) the set of annotated \( m \)-sketches of size \( n \). It is easy to see that a word \( \hat{w} = \hat{w}_1 \cdots \hat{w}_{(m+1)n} \) is in \( D^{(m)}(n) \) if and only if

(a) \( \{ \hat{w}_1, \ldots, \hat{w}_{(m+1)n} \} = A^{(m)}(n) \),
(b) for all \( i \in [n] \) and all \( s \in [m] \), the letter \( \alpha_i^{(s-1)} \) appears before \( \alpha_i^{(s)} \),
(c) for all \( i, j \in [n] \) and all \( s, t \in [m] \), if \( \alpha_i^{(s-1)} \) appears before \( \alpha_j^{(t-1)} \), then \( \alpha_i^{(s)} \) appears before \( \alpha_j^{(t)} \).

![Diagram](image_url)

**Figure 9.** The mapping \( \sigma_1 \) associating an annotated 1-sketch to any point \((x_1, \ldots, x_n)\) in \( \mathbb{R}^n \setminus \bigcup_{H \in A_{[-m,m]}(n)} H \). Graphically, the annotated 1-sketch \( \hat{w} = \hat{w}_1 \cdots \hat{w}_{2n} \) is represented by a set of non-nesting parentheses on \( 2n \)-points corresponding to the letters. More precisely, if the letters \( \alpha_i^{(0)} \) to \( \alpha_i^{(1)} \) are in position \( p \) and \( q \) of \( \hat{w} \), then parenthesis labeled \( i \) goes from the \( p \)th point to the \( q \)th point.

We now associate an annotated \( m \)-sketch of size \( n \) to each region of \( A_{[-m,m]}(n) \). Let \( x = (x_1, \ldots, x_n) \) be a point in \( \mathbb{R}^n \setminus \bigcup_{H \in A_{[-m,m]}(n)} H \). Observe that the condition \( x \notin \bigcup_{H \in A_{[-m,m]}(n)} H \) is equivalent to the fact that the numbers \( \{ x_i + s \mid i \in [n], s \in [0..m] \} \) are all distinct. We define \( z_1, \ldots, z_{(m+1)n} \) by the conditions \( z_1 < \ldots < z_{(m+1)n} \) and \( \{ z_1, \ldots, z_{(m+1)n} \} = \{ x_i + s \mid i \in [n], s \in [0..m] \} \). Then, we define \( \sigma_m(x) = \hat{w}_1 \hat{w}_2 \cdots \hat{w}_{(m+1)n} \), where \( \hat{w}_p = \alpha_i^{(s)} \) if \( z_p = x_i + s \). Here are basic properties of the mapping \( \sigma_m \).

(i) For any \( x \notin \bigcup_{H \in A_{[-m,m]}(n)} H \), the word \( \sigma_m(x) \) is an annotated \( m \)-sketch. Indeed, it clearly satisfies the conditions (a), (b), (c).

(ii) The mapping \( \sigma_m \) is constant over each region of \( A_{[-m,m]}(n) \). Indeed, the order of the numbers \( \{ x_i + s \mid i \in [n], s \in [0..m] \} \) cannot change when \( x \) moves continuously inside \( \mathbb{R}^n \setminus \bigcup_{H \in A_{[-m,m]}(n)} H \).

(iii) The sketch \( \sigma_m(x) \) identifies the region containing \( x \). Indeed, for all \( i, j \in [n] \), and all \( s \in [0..m] \), \( x_i - x_j < s \) if \( \alpha_i^{(0)} \) appears before \( \alpha_j^{(s)} \) in \( \sigma_m(x) \) and \( x_i - x_j > s \) otherwise.
(iv) For any annotated \( m \)-sketch \( \hat{w} = \hat{w}_1 \ldots \hat{w}_{(m+1)n} \), there exists \( x \in \mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}_{[-m,m]}(n)} H \) such that \( \sigma(x) = \hat{w} \). Indeed, one can simultaneously define \( x \in \sigma^{-1}(\hat{w}) \) and \( z_1, \ldots, z_{(m+1)n} \) by applying the following rule for \( p = 1, 2, \ldots, (m + 1)n \): if \( \hat{w}_p = \alpha_i^{(0)} \) then set \( z_p = z_{p-1} + 1/(n+1) \) and \( x_i = z_p \), while if \( \hat{w}_p = \alpha_i^{(s)} \) with \( s \neq 0 \) then set \( z_p = x_i + s \).

Properties (i) and (ii) show that \( \sigma_m \) is a mapping from the regions of \( \mathcal{A}_{[-m,m]}(n) \) to \( \mathcal{D}^{(m)}(n) \). Properties (iii) and (iv) imply that \( \sigma_m \) is a bijection. In particular, this shows that \( \mathcal{A}_{[-m,m]}(n) \) has \( \binom{(m+1)n}{(mn+1)/2} \) regions. The bijection \( \sigma_1 \) is represented in Figure 10.

Note that (iii) gives the inverse bijection \( \sigma_m^{-1} \) in terms of inequality, while (iv) gives an explicit point in \( \sigma_m^{-1}(\hat{w}) \).

Next, we describe a bijection \( \hat{\phi}_m \) between \( \mathcal{D}^{(m)}(n) \) and the set \( \mathcal{T}^{(m)}(n) \) of \((m+1)\)-ary trees with labeled nodes.\(^3\)

Let \( T \) be a rooted plane tree. We define the drift of a vertex \( v \) of \( T \) as drift\((v) = \text{lsib}(u_0) + \cdots + \text{lsib}(u_k) \), where \( u_0, u_1, \ldots, u_k = v \) are the vertices on the path from the root \( u_0 \) to \( v \). We define the total order \( \prec_T \) on vertices of \( T \) by setting \( u \prec_T v \) if either \( \text{drift}(u) < \text{drift}(v) \), or \( \text{drift}(u) = \text{drift}(v) \) and \( u \) appears before \( v \) in the postfix order of \( T \) (recall that the postfix order is the order of appearance of the vertices when turning counterclockwise around the tree starting at the root). The order \( \prec_T \) is represented in Figure 11(a).

Let \( \hat{T}(n) \) be the set of rooted plane trees, with (at most \( n \)) nodes labeled with distinct numbers in \([n] \), and some special leaves called buds (see Figure 12). If \( T \) has some buds, we call first bud the least bud of \( T \) for the order \( \prec_T \). Let \( \hat{w} = \hat{w}_1 \ldots \hat{w}_{(m+1)n} \in \mathcal{D}^{(m)}(n) \), and let \( \hat{w}^p = \hat{w}_1 \ldots \hat{w}_p \) be its prefix of length \( p \). We define the trees \( \hat{\phi}_m(\hat{w}^0), \ldots, \hat{\phi}_m(\hat{w}^{(m+1)n}) \in \hat{T}(n) \) as follows:

- \( \hat{\phi}_m(\hat{w}^0) \) is the tree with one bud and no other vertex,
- if \( p > 0 \) and \( \hat{w}_p = \alpha_i^{(s)} \) with \( s > 0 \), then \( \hat{\phi}_m(\hat{w}^p) \) is obtained from \( \hat{\phi}(\hat{w}^{p-1}) \) by replacing its first bud by a (non-bud) leaf,
- if \( p > 0 \) and \( \hat{w}_p = \alpha_i^{(0)} \), then \( \hat{\phi}_m(\hat{w}^p) \) is obtained from \( \hat{\phi}(\hat{w}^{p-1}) \) by replacing its first bud by a node labeled \( i \) with \( m+1 \) children, all of them buds.

The trees \( \hat{\phi}(\hat{w}^p) \) are represented in Figure 12. It is clear, by induction on \( p \), that \( \hat{\phi}_m(\hat{w}^p) \) has \( 1 + mn_\alpha - n_\beta \) buds, where \( n_\alpha \) and \( n_\beta \) are respectively the number of \( \alpha \)-letters and \( \beta \)-letters in \( \hat{w}^p \). In particular \( \hat{\phi}_m(\hat{w}^p) \) has at least one bud, so that \( \hat{\phi}_m \) is well defined, and \( \hat{\phi}_m(\hat{w}) \) has exactly one bud. We denote \( \phi_m(\hat{w}) \) the tree in \( \mathcal{T}^{(m)}(n) \) obtained from \( \hat{\phi}_m(\hat{w}) \) by replacing its bud by a leaf; see Figure 11(b). Before showing that \( \phi_m \) is a bijection, we describe the inverse mapping \( \psi_m \). Let \( T \in \mathcal{T}^{(m)}(n) \) and let \( u_0 \prec_T u_1 \prec_T \cdots \prec_T u_{(m+1)n} \) be the vertices of \( T \) (\( T \) has \( n \) nodes and \( mn + 1 \) leaves).

We denote \( \psi_m(T) \) the word \( \hat{w} = \hat{w}_1 \ldots \hat{w}_{(m+1)n} \) defined as follows: for all \( p \in [(m+1)n] \), if \( u_p \) is the \((s+1)\)st child of the node \( i \), then \( \hat{w}_p = \alpha_i^{(s)} \).

\(^3\)Of course, any classical bijection between \( m \)-parenthesis systems and \((m+1)\)-ary trees induces a bijection between \( \mathcal{D}^{(m)}(n) \) and \( \mathcal{T}^{(m)}(n) \) by sending labels from the parentheses to the nodes. The non-classical bijection \( \phi_m \) is chosen because it is well adapted to the “non-nesting” nature of annotated sketches.
Figure 10. The Catalan arrangement $A_{\{-1,0,1\}}(3)$, and the annotated 1-sketches corresponding to each region. The sketches marked $A$ are Shi maximal, the sketches marked $B$ are semiorder maximal, and those marked both $A$ and $B$ are Linial maximal.

**Proposition 8.1.** The mapping $\phi_m$ is a bijection between $D^{(m)}(n)$ and $T^{(m)}(n)$. The inverse mapping is $\psi_m$. Moreover, for $\hat{w} \in D^{(m)}(n)$, if the letter following $a_i^{(s)}$ in $\hat{w}$ is $a_j^{(t)}$, then the $(s+1)$st child of the node $i$ in $T = \phi_m(\hat{w})$ is the node $j$ if $t = 0$, and a leaf otherwise.

**Proof.** First note that for all $T \in T^{(m)}(n)$, the word $\psi_m(T)$ clearly satisfies the properties (a), (b), (c) of annotated $m$-sketches. Hence $\psi_m$ is a mapping from $T^{(m)}(n)$ to $D^{(m)}(n)$. 
Next we give an alternative description of $\psi_m$. Let $T \in T^{(m)}(n)$, and let $u_0 \prec_T u_1 \prec_T \cdots \prec_T u_{(m+1)n}$ be its vertices. We denote $\hat{\psi}_m(T)$ the word $\hat{w} = \hat{\psi}_m(T)$, where $\hat{w}_p = \alpha_i$ if $u_{p-1}$ is the node labeled $i$, and $\hat{w}_p = \beta$ if $u_{p-1}$ is a leaf. We now show that $\hat{w} = \psi_m(T)$ is the annotation of $\hat{w} = \hat{\psi}_m(T)$. It is easy to see that for all $q \in [0..(m+1)n]$, if the vertex $u_q$ is a node, then $u_{q+1}$ is its first child. Now let $p \in [(m+1)n]$ such that $\hat{w}_p = \alpha_i^{(0)}$ for some $i \in [n]$. In this case $u_p$ is the first child of the node $i$, hence $u_{p-1}$ is the node $i$ and $\hat{w}_p = \alpha_i$. Suppose now that $p \in [(m+1)n]$ is such that $\hat{w}_p = \alpha_i^{(s)}$ for some $s \in [m], i \in [n]$. In this case $u_p$ is not a first child, thus $u_{p-1}$ is a leaf and $\hat{w}_p = \beta$. This proves that $\hat{w} = \psi_m(T)$ is indeed the annotation of $\hat{w} = \hat{\psi}_m(T)$.
Next we prove that $\psi_m \circ \phi_m = \text{Id}$ and $\phi_m \circ \psi_m = \text{Id}$. Let $\hat{w} \in T^{(m)}(n)$, and let $\hat{w}^p$ be the prefix of length $p$. Let $T = \phi(\hat{w})$, and let $T^p = \phi_m(\hat{w}^p)$. We claim that the order in which the non-leaf vertices are created in the sequence $\phi_m(\hat{w}^0), \phi_m(\hat{w}^1), \ldots, \phi_m(\hat{w}^m), \phi_m(\hat{w})$ is the same as the order $\prec_T$. Indeed, for all $p \in [(m+1)n]$ the order $\prec_{T^p}$ on the vertices of $T^p$ coincide with the order $\prec_T$. So the first bud of $T^p$ is less than any bud of $T^{p+1}$ for the order $\prec_T$. This establish the claim. From the claim, and the alternative description of $\psi_m$ it follows directly that $\psi_m \circ \phi_m(\hat{w}) = \hat{w}$. And since $|D^{(m)}(n)| = |T^{(m)}(n)|$, $\phi_m$ and $\psi_m$ are inverse bijections.

Lastly, suppose that for $\hat{w} \in D^{(m)}(n)$ we have $\hat{w}_p = \alpha_i^{(s)}$ and $\hat{w}_{p+1} = \alpha_j^{(t)}$. In this case, $u_p$ is the $(s + 1)st$ child of the node $i$ (by definition of $\psi_m$) and $u_p$ is the node $j$ if $s = 0$ and a leaf otherwise (by definition of $\psi_m$).

We denote $\Phi_m = \phi_m \circ \sigma_m$ the bijection from the the regions of $A_{[-m..m]}(n)$ to $T^{(m)}(n)$, and $\Psi_m = \sigma_m^{-1} \circ \psi_m$ its inverse.

**Lemma 8.2.** For $T \in T^{(m)}(n)$, $\Psi_m(T)$ is the region of $A_{[-m..m]}(n)$ made of the points $x = (x_1, \ldots, x_n)$ satisfying the following inequalities for all distinct integers $i, j \in [n]$, and all $s \in [0..m]$:
\[
x_i - x_j < s \text{ if } i \prec_T v \text{ where } v \text{ is the } (s + 1)st \text{ child of the node } j, \text{ and } x_i - x_j > s \text{ otherwise}.
\]

**Proof.** Let $x \in \Psi_m(T) = \psi_m(\sigma_m^{-1}(T))$. By Property (iii) of $\sigma_m$, $x_i - x_j < s$ if and only if $\alpha_i^{(0)}$ appears before $\alpha_j^{(s)}$ in $\hat{w} = \psi_m(T)$. By definition of $\psi_m$, this happens if and only if $u \prec_T v$, where $u$ is the first child of the node $i$. Moreover, since $u \neq v$ and $u$ is the successor of $i$ in the $\prec_T$ order, this happens if and only if $i \prec_T v$. \[\square\]

### 8.2. Bijections for the Shi, semiorder, and Linial arrangements.

In this section we give a bijection between the regions of $A_S(n)$, and the trees in $T_S$ for all $S \subseteq \{-1, 0, 1\}$. Up to symmetry, the interesting cases are $S = \{-1, 0, 1\}$ (Catalan arrangement), $S = \{0, 1\}$ (Shi arrangement), $S = \{-1, 1\}$ (semiorder arrangement), $S = \{1\}$ (Linial arrangement), and $S = \{0\}$ (braid arrangement). We already treated the case $S = \{-1, 0, 1\}$ in the previous Section. For the Shi arrangements our bijection can be seen as a relative to [7] as discussed in Section 9.1. The bijection for the semiorder and Linial arrangements seem to be new.

The basic idea of our bijection for the Shi, semiorder, and Linial arrangements is to think of regions in these arrangements as union of regions of the Catalan arrangement. Then we will choose a canonical representative among these regions, so as to identify regions of the Shi, semiorder and Linial arrangements with certain canonical 1-sketches. We show that the bijection $\phi_1$ between $D^{(1)}(n)$ and $T^{(1)}(n)$ induces bijections between the canonical 1-sketches for $A_{\{0,1\}}(n)$, $A_{\{-1,1\}}(n)$, and $A_{\{1\}}(n)$ and the trees in $T_{\{0,1\}}(n)$, $T_{\{-1,1\}}(n)$, and $T_{\{1\}}(n)$ respectively. This is represented in Figure 13. Moreover, the bijections induced by $\Phi_1 = \phi_1 \circ \sigma_1$ between regions and trees have simple inverses.

**Definition 8.3.** Let $\hat{w}$ and $\hat{w}'$ be annotated 1-sketches of size $n$. We say that $\hat{w}$ and $\hat{w}'$ are related by a Shi move if $\hat{w}'$ is obtained from $\hat{w}$ by inverting two consecutive letters $\alpha_i^{(1)}$ and $\alpha_j^{(0)}$ with $i < j$. We say that $\hat{w}$ and $\hat{w}'$ are related by a semiorder move if $\hat{w}'$ is obtained from $\hat{w}$ by inverting two consecutive letters $\alpha_i^{(0)}$ and $\alpha_j^{(0)}$ and also two
consecutive letters $\alpha_i^{(1)}$ and $\alpha_j^{(1)}$ (for the same pair $\{i, j\}$). We say that $\hat{w}$ and $\hat{w}'$ are related by a Linial move if they are related by either a Shi or semiorder move. Lastly, we say that $\hat{w}$ and $\hat{w}'$ are Shi equivalent (resp. semiorder equivalent, Linial equivalent) if one can be obtained from the other by performing a series of Shi (resp. semiorder, Linial) moves.

Let $\hat{w}, \hat{w}' \in D^{(1)}(n)$ be annotated 1-sketches. Let $\rho = \sigma^{-1}(\hat{w})$ and $\rho' = \sigma^{-1}(\hat{w}')$ be the regions of $A_{(-1,0,1)}(n)$ corresponding to $\hat{w}$ and $\hat{w}'$. Observe that $\hat{w}$ and $\hat{w}'$ are related by the Shi move inverting $\alpha_i^{(0)}$ and $\alpha_j^{(1)}$ if and only if the regions $\rho$ and $\rho'$ are separated only by the hyperplane $H_{i,j,-1} = \{x_i - x_j = -1\}$. Thus, $\hat{w}$ and $\hat{w}'$ are Shi equivalent if and only if one can go from $\rho$ to $\rho'$ only crossing hyperplanes of the forms $H_{i,j,-1}$ for $i < j$. In other words, $\hat{w}$ and $\hat{w}'$ are Shi equivalent if and only if $\rho$ and $\rho'$ are contained in the same region of the Shi arrangement $A_{(0,1)}(n)$. Similarly, $\hat{w}$ and $\hat{w}'$ are related by the semiorder move inverting $\alpha_i^{(0)}$ and $\alpha_j^{(1)}$ (and $\alpha_i^{(1)}$ and $\alpha_j^{(0)}$) if and only if the regions $\rho$ to $\rho'$ are separated only by the hyperplane $H_{i,j,0} = \{x_i - x_j = 0\}$. Thus, $\hat{w}$ and $\hat{w}'$ are semiorder equivalent if and only if $\rho$ and $\rho'$ are contained in the same region of the semiorder arrangement $A_{(-1,1)}(n)$. Also, $\hat{w}$ and $\hat{w}'$ are Linial equivalent if and only if $\rho$ and $\rho'$ are contained in the same region of the Linial arrangement $A_{(1)}(n)$.

To summarize:

**Lemma 8.4.** Let $\hat{w}$ and $\hat{w}'$ be annotated 1-sketches of size $n$, and let $\rho = \sigma^{-1}(\hat{w})$ and $\rho' = \sigma^{-1}(\hat{w}')$ be the regions of $A_{(-1,0,1)}(n)$ corresponding to $\hat{w}$ and $\hat{w}'$. The annotated 1-sketches $\hat{w}$ and $\hat{w}'$ are Shi (resp. semiorder, Linial) equivalent if and only if $\rho$ and $\rho'$ are contained in the same region of $A_{(0,1)}(n)$ (resp. $A_{(-1,1)}(n)$, $A_{(1)}(n)$).

We consider the lexicographic order $< \in D^{(1)}(n)$ given by the following order on the alphabet: $\alpha_1^{(1)} < \alpha_2^{(1)} < \cdots < \alpha_n^{(1)} < \alpha_1^{(0)} < \alpha_2^{(0)} < \cdots < \alpha_n^{(0)}$. We say that an annotated 1-sketch $\hat{w}$ is Shi locally-maximal (resp. semiorder locally-maximal, Linial locally-maximal) if it is larger than any 1-sketch obtained from $\hat{w}$ by a single Shi (resp. semiorder, Linial) move. We say that an annotated 1-sketch $\hat{w}$ is Shi maximal (resp. semiorder maximal, Linial maximal) if it is larger than any Shi (resp. semiorder, Linial) equivalent 1-sketch. The maximal 1-sketches are indicated in Figure 10.

On the one hand, Lemma 8.4 implies that regions of $A_{(0,1)}(n)$ (resp. $A_{(-1,1)}(n)$, $A_{(1)}(n)$) are in bijection with Shi (resp. semiorder, Linial) maximal 1-sketches in $D^{(1)}(n)$. On the other hand, locally-maximal 1-sketches are easy to characterize. The following result shows that the two notions actually coincide.

**Lemma 8.5.** An annotated 1-sketch $\hat{w} \in D^{(1)}(n)$ is Shi (resp. semiorder, Linial) maximal if and only if it is Shi (resp. semiorder, Linial) locally-maximal.

Before proving Lemma 8.5 we explore its consequences.

**Corollary 8.6.** The mapping $\Psi_1 = \Phi_1^{-1}$ between $T^{(1)}(n)$ and the regions of $A_{(-1,0,1)}(n)$ induces a bijection $\Psi_{[0,1]}$ (resp. $\Psi_{(-1,1)}$, $\Psi_{(1)}$) between the trees in $T_{[0,1]}(n)$ (resp. $T_{(-1,1)}(n)$, $T_{(1)}(n)$) and the regions of $A_{(0,1)}(n)$ (resp. $A_{(-1,1)}(n)$, $A_{(1)}(n)$).

(1) For $T \in T_{[0,1]}(n)$ the region $\Psi_{[0,1]}(T)$ is defined by the following inequalities for all $1 \leq i < j \leq n$: $x_i - x_j < 0$ if $i < T j$ (that is to say, node $i$ is less than
easy to see (by induction on the number of Shi moves), that this letter is necessarily \( Ψ \) and ˆφ inequalities defining the region \( Ψ \) if and only if no node with \( j > i \) is Shi locally-maximal if and only if \( φ \) is Shi locally-maximal and semiorder locally-maximal. Thus ˆφ is Shi locally-maximal if and only if for all \( i \leq j \leq n \): \( x_i - x_j < 1 \) iff \( i \prec_T v \), where \( u \) is the right child of \( i \) and \( v \) is the right child of \( j \).

(3) For \( T \in \mathcal{T}_{\{0,1\}}(n) \) the region \( Ψ_{\{1\}}(T) \) is defined by the following inequalities for all \( 1 \leq i < j \leq n \): \( x_i - x_j < 1 \) iff \( i \prec_T v \), where \( v \) is the right child of \( j \).

Corollary 8.6 is illustrated in Figure 13.

**Remark 8.7.** The method used above works just as well for the braid arrangement: the induced bijection is between the regions of \( A_{\{0\}}(n) \) and the binary trees with no right child. These trees (which look like paths) are the ones near the origin in Figure 13 (note that they are a subsets of the Shi trees). Of course, such a machinery is unnecessary for this simple case, which could be treated with \( m = 0 \), but it illustrates the fact that arrangements corresponding to small values of \( m \) are embedded in the bijective framework corresponding to larger values of \( m \).

**Proof of Corollary 8.6.** It is clear that an annotated 1-sketch \( \hat{w} \) is Shi locally-maximal if and only if for all \( i \in [n] \) the letter \( α_i^{(1)} \) is not followed by a letter \( α_j^{(0)} \) with \( j > i \). By Proposition 8.1 this means that \( \hat{w} \) is Shi locally-maximal if and only if for all \( i \in [n] \) the right child of the node \( i \) in \( T = φ_1(\hat{w}) \) is not a node \( j \) with \( j > i \). In other words, \( \hat{w} \) is Shi locally-maximal if and only if \( φ_1(\hat{w}) \) is in \( \mathcal{T}_{\{0,1\}} \).

Similarly, an annotated 1-sketch \( \hat{w} \) is semiorder locally-maximal if and only if for all \( i \in [n] \) the letters \( α_i^{(0)} \) and \( α_i^{(1)} \) are not followed by the letters \( α_j^{(0)} \) and \( α_j^{(1)} \) respectively, with \( j > i \). Note that if \( α_i^{(0)} \) is followed by \( α_j^{(0)} \) and \( α_i^{(1)} \) is followed by a \( β \)-letter, then this letter is necessarily \( α_j^{(1)} \). Thus, by Proposition 8.1 \( \hat{w} \) is semiorder locally-maximal if and only if no node \( i \in [n] \) of \( T \) has both a left child which is a node \( j > i \) and a right child which is a leaf. Thus, \( \hat{w} \) is semiorder locally-maximal if and only if \( φ_1(\hat{w}) \) is in \( \mathcal{T}_{\{0,1\}} \).

Lastly, an annotated 1-sketch \( \hat{w} \) is Linial locally-maximal if and only if it is both Shi locally-maximal and semiorder locally-maximal. Thus \( \hat{w} \) is Linial locally-maximal if and only if \( φ_1(\hat{w}) \) is in \( \mathcal{T}_{\{0,1\}} \cap \mathcal{T}_{\{0,1\}} = \mathcal{T}_{\{1\}} \).

Moreover, the description of the bijection \( Ψ_S \) is immediate from Lemma 8.2 as the inequalities defining the region \( Ψ_S(T) \) are a subset of the inequalities defining the region \( Ψ_1(T) \) (the inequalities of the form \( x_i - x_j < s \) or \( x_i - x_j > s \) for \( i < j \) and \( s \in S \)).

**Proof of Lemma 8.5.** We first treat the case of the Shi arrangement. Let \( \hat{w}_1 = \hat{w}_1 \cdots \hat{w}_{2n} \) and \( \hat{w}' = \hat{w}'_1 \cdots \hat{w}'_{2n} \) be two 1-sketches. Suppose that \( \hat{w} \) and \( \hat{w}' \) are Shi equivalent. It is easy to see (by induction on the number of Shi moves), that

(a) for all \( i, j \in [n] \), \( α_i^{(0)} \) appears before \( α_j^{(0)} \) in \( \hat{w} \) if and only if \( α_i^{(0)} \) appears before \( α_j^{(0)} \) in \( \hat{w}' \),

(b) for all \( i > j \in [n] \), \( α_i^{(1)} \) appears before \( α_j^{(0)} \) in \( \hat{w} \) if and only if \( α_i^{(1)} \) appears before \( α_j^{(0)} \) in \( \hat{w}' \).
Figure 13. The Catalan arrangement $\mathcal{A}_{\{1,0,1\}}(3)$, and the labeled binary trees corresponding to each region. The trees marked $A$ (including those denoted $A+B$) are in bijection with the regions of the Shi arrangement $\mathcal{A}_{\{0,1\}}(3)$. The trees marked $B$ (including those denoted $A+B$) are in bijection with the regions of the semiorder arrangement $\mathcal{A}_{\{-1,1\}}(3)$. The trees marked $A+B$ are in bijection with the regions of the Linial arrangement $\mathcal{A}_{\{1\}}(3)$.

Now suppose that $\hat{w}$ is Shi locally-maximal and $\hat{w}'$ is Shi maximal. We want to show $\hat{w} = \hat{w}'$. Suppose by contradiction that they are different, and let $p \in [2n]$ be such that $\hat{w}_p \neq \hat{w}'_p$ and $\hat{w}_k = \hat{w}'_k$ for all $k \in [p-1]$. Since $\hat{w} < \hat{w}'$, either

(i) $\hat{w}_p = \alpha_i^{(1)}$ and $\hat{w}'_p = \alpha_j^{(1)}$ with $i < j$,

(ii) $\hat{w}_p = \alpha_i^{(0)}$ and $\hat{w}'_p = \alpha_j^{(0)}$ with $i < j$, 


(iii) \( \hat{w}_p = \alpha_i^{(1)} \) and \( \hat{w}'_p = \alpha_j^{(0)} \) for some \( i, j \).

However case (i) is impossible for 1-sketches: if \( \hat{w}_k = \hat{w}'_k \) for all \( k \in [p-1] \), \( \hat{w}_p = \alpha_i^{(1)} \) and \( \hat{w}'_p = \alpha_j^{(1)} \) then \( i = j \). Moreover case (ii) is impossible by (a). Hence (iii) holds.

Let \( q > p \) be such that \( \hat{w}_q = \alpha_j^{(0)} \). By (a) and (b), we must have \( \hat{w}_{q-1} = \alpha_k^{(1)} \) with \( k < j \). But this contradicts the fact that \( \hat{w} \) is Shi locally-maximal. Hence \( \hat{w} = \hat{w}' \) as wanted.

Next, we treat the case of the semiorder arrangement. Given \( \hat{w} \in D^1(n) \), we say that \( i, j \in [n] \) are \( \hat{w} \)-exchangeable if in \( \hat{w} \) the letters \( \alpha_i^{(0)} \) and \( \alpha_j^{(0)} \) are separated only by \( \alpha \)-letters, and \( \alpha_i^{(1)} \) and \( \alpha_j^{(1)} \) are separated only by \( \beta \)-letters. Let \( \hat{w}, \hat{w}' \in D^1(n) \) be two semiorder equivalent 1-sketches. It is easy to see that \( \hat{w}' \) is obtained from \( \hat{w} \) by replacing the letters \( \alpha_i^{(0)} \) and \( \alpha_i^{(1)} \) by \( \alpha_i^{(0)} \) and \( \alpha_i^{(1)} \) for a permutation \( \pi \) of \( [n] \) such that for all \( i \in [n] \), \( i \) and \( \pi(i) \) are \( \hat{w} \)-exchangeable. Now suppose that \( \hat{w} \) is semiorder locally-maximal and \( \hat{w}' \) is semiorder maximal. We want to show \( \hat{w} = \hat{w}' \). Suppose by contradiction that they are different, and let \( p \in [2n] \) be such that \( \hat{w}_p \neq \hat{w}'_p \) and \( \hat{w}_k = \hat{w}'_k \) for all \( k \in [p-1] \). Since \( \hat{w} < \hat{w}' \), either (i), (ii), or (iii) holds. However (i) is impossible as before, and (iii) is impossible because the parenthesis systems underlying \( \hat{w} \) and \( \hat{w}' \) are equal. Hence, (ii) holds. By the remark above, \( i, j \) are \( \hat{w} \)-equivalent.

Hence denoting \( \hat{w}_{p+d} = \alpha_j^{(0)} \), we get that for all \( d \in [d] \) the letter \( \hat{w}_{p+d} \) has the form \( \alpha_{i_c}^{(0)} \) for some \( i_c \) which is \( \hat{w} \)-exchangeable with \( i \). Since \( \hat{w} \) is locally maximal, we have \( i > i_1 > \cdots > i_d = j \). This contradicts \( i < j \), hence \( \hat{w} = \hat{w}' \) as wanted.

Lastly, we treat the case of the Linial arrangement. Let \( \hat{w}, \hat{w}' \in D^1(n) \) be two Linial equivalent 1-sketches. It is easy to see that (b) holds. Suppose now that \( \hat{w} \) is Linial locally-maximal and \( \hat{w}' \) is Linial maximal. We want to show \( \hat{w} = \hat{w}' \). Suppose by contradiction that they are different, and let \( p \in [2n] \) be such that \( \hat{w}_p \neq \hat{w}'_p \) and \( \hat{w}_k = \hat{w}'_k \) for all \( k \in [p-1] \). Since \( \hat{w} < \hat{w}' \), either (i), (ii), or (iii) holds. However (i) is impossible as before, so that \( \hat{w}_p = \alpha_j^{(0)} \) for some \( j \in [n] \). Let \( d > 0 \) such that \( \hat{w}_{p+d} = \alpha_j^{(0)} \).

Suppose first that \( \hat{w}_p, \hat{w}_{p+1}, \ldots, \hat{w}_{p+d} \) are all \( \alpha \)-letters. We denote \( \hat{w}_{p+c} = \alpha_c^{(0)} \) for all \( c \in [d] \). In this case, \( i_0 < i_d = j \) (since \( \hat{w} < \hat{w}' \)), hence taking the least index \( i_c \) we have \( i_c < i_{c+1} \) and \( i_c < j \). Since \( \hat{w} \) is Linial locally-maximal, the letter following \( \alpha_i^{(1)} \) has the form \( \alpha_k^{(0)} \) with \( k < i_c \) (otherwise it would be \( \alpha_{i_{c+1}}^{(1)} \) or \( \alpha_k^{(0)} \) with \( k > i_c \) and one could do an increasing Linial move) Lastly, since \( k < i_c \), \( j \) and \( \alpha_k^{(0)} \) is between \( \alpha_i^{(1)} \) and \( \alpha_j^{(1)} \) in \( \hat{w} \), property (b) implies that \( \alpha_k^{(0)} \) is between \( \alpha_{i_c}^{(1)} \) and \( \alpha_j^{(1)} \) in \( \hat{w}' \). Hence \( \alpha_{i_c}^{(0)} \) appears before \( \alpha_j^{(0)} \) in \( \hat{w}' \). We reach a contradiction. It remains to treat the case where \( \{ \hat{w}_p, \hat{w}_{p+1}, \ldots, \hat{w}_{p+d-1} \} \) contains a \( \beta \)-letter. Let \( \alpha_i^{(1)} \) be the last \( \beta \)-letter before \( \alpha_j^{(0)} \) in \( \hat{w} \), and let \( \alpha_{i_0}^{(0)} \) be the letter following \( \alpha_i^{(1)} \). By (b), we have \( i < j \). Moreover, since \( \hat{w} \) is Linial locally-maximal, \( i_0 < i \). Since \( i_0 < j \) and all the letters between \( \alpha_{i_0}^{(0)} \) and \( \alpha_j^{(0)} \) are \( \alpha \)-letters, the same reasoning as before leads to a contradiction. Hence \( \hat{w} = \hat{w}' \) as wanted. \( \square \)
8.3. General bijection for transitive deformation of the braid arrangement.

In this section we generalize the strategy adopted in Section 8.2 in order to establish bijections between regions of arbitrary transitive deformations of the braid arrangement, and trees.

We fix a positive integer $N$ and a $\binom{N}{2}$-tuple of finite sets of integers $S = (S_{i,j})_{1 \leq i < j \leq N}$. Recall that $A_S$ is the arrangement in $\mathbb{R}^N$ made of the hyperplanes $H_{i,j,s}$ for all $1 \leq i < j \leq N$ and all $s \in S_{i,j}$. Recall also that when $S$ is transitive, the regions of $A_S$ are equinumerous to the trees $T_S$ defined in Definition 4.3. For $T \in T_S$, we denote $\Psi_S(T)$ the set of points $(x_1, \ldots, x_N)$ satisfying the following inequalities for all $1 \leq i < j \leq N$ and $s \in S_{i,j}$:

- for $s \geq 0$, $x_i - x_j < s$ if the node $i$ is less than the $(s+1)$st child of the node $j$ in the $\prec_T$ order, and $x_i - x_j > s$ otherwise,
- for $s < 0$, $x_i - x_j > s$ if the node $j$ is less than the $(-s+1)$st child of the node $i$ in the $\prec_T$ order, and $x_i - x_j < s$ otherwise.

Our goal is to establish the following result.

**Theorem 8.8.** If $S = (S_{i,j})_{1 \leq i < j \leq N}$ is transitive (see Definition 4.3), then $\Psi_S$ is a bijection between the set $T_S$ of trees and the regions of $A_S$.

**Remark 8.9.** The bijections $\Psi_S$ are compatible with refinements of arrangements. Indeed $A'_S$ is a refinement of $A_S$ if and only if $S' = (S'_{i,j})_{1 \leq i < j \leq N}$, with $S_{i,j} \subseteq S'_{i,j}$ for all $i, j$. In this case, $T_S \subseteq T'_S$, and for all $T \in T_S$, $\Psi_S(T) \subseteq \Psi_{S'}(T)$.

Our strategy to prove Theorem 8.8 is the same as in Section 8.2. Let $m = \max(|s|, s \in \cup S_{i,j})$, so that $T_S$ is a subarrangement of $A_{[-m..m]}(N)$. We will think of regions of $A_S$ as equivalence class of regions of $A_{[-m..m]}(N)$, and the bijection $\Phi_m$ defined in Section 8.1 will induce a bijection $\Phi_S$ between regions of $A_S$ and $T_S$.

**Definition 8.10.** Let $\hat{w}, \hat{w}'$ be annotated $m$-sketches of size $N$. Let $i,j \in [N]$ with $i < j$, and let $s \in [-m..m]$. We say that $\hat{w}$ and $\hat{w}'$ are related by a $(i,j,s)$-move if for all $k \in [0..m] \cap [-s..m - s]$ the pair of letters $\{\alpha_i^{(k)}, \alpha_j^{(s+k)}\}$ are consecutive in $\hat{w}$, and $\hat{w}'$ is obtained from $\hat{w}$ by inverting each of these pairs. A $S$-move is any $(i,j,s)$-move with $1 \leq i < j \leq N$, and $s \notin S_{i,j}$. We say that $\hat{w}$ and $\hat{w}'$ are $S$-equivalent if one can be obtained from the other by performing a series of $S$-moves.

**Example 8.11.** Let $m = 1$. The Shi moves defined in Section 8.2 are all the $(i,j,-1)$-moves, and the semiorder moves are all the $(i,j,0)$-moves. Hence the Shi (resp. semiorder, Linial) moves are the $S$-moves for the tuple $S = (S_{i,j})_{1 \leq i < j \leq N}$ with $S_{i,j} = \{0, 1\}$ (resp. $S_{i,j} = \{-1, 1\}$, $S_{i,j} = \{1\}$) for all $1 \leq i < j \leq N$.

We consider the following order $\prec$ on the alphabet $A^{(m)}(N)$: $\alpha_i^{(s)} \prec \alpha_j^{(t)}$ if either $s > t$, or $s = t$ and $i < j$. We now consider the lexicographic order $\prec$ on $D^{(m)}(n)$ corresponding the order $\prec$ on the letters. An annotated $m$-sketch $\hat{w} \in D^{(m)}(n)$ is $S$-locally-maximal if it is greater than any $m$-sketch obtained from $\hat{w}$ by a single $S$-move. It is $S$-maximal if it is greater than any $S$-equivalent $m$-sketch. Lastly, a region $\rho$ of $A_{[-m..m]}(N)$ is said $S$-maximal if the annotated $m$-sketch $\sigma_m(\rho)$ is $S$-maximal. We now establish two easy lemmas.

**Lemma 8.12.** Each region of $A_S$ contains a unique $S$-maximal region of $A_{[-m..m]}(N)$. 
Proof. Let \( \hat{w}, \hat{w}' \in D^{(m)}(N) \), and let \( \rho = \sigma^{-1}_m(\hat{w}) \) and \( \rho' = \sigma^{-1}_m(\hat{w}') \) be the associated regions of \( A_{[-m..m]}(n) \). It is clear that \( \hat{w} \) are \( \hat{w}' \) are related by a \((i, j, s)\)-move (for \( 1 \leq i < j \leq N \), and \( s \in [-m..m] \)) if and only if the regions \( \rho \) and \( \rho' \) are separated only by the hyperplane \( H_{i,j,s} \). Thus \( \hat{w} \) and \( \hat{w}' \) are \( S \)-equivalent if and only if \( \rho \) and \( \rho' \) are in the same regions of \( A_S \). Thus, for in any region \( R \) of \( A_S \), exactly one of the regions \( \rho \) of \( A_{[-m..m]}(N) \) contained in \( R \) is \( S \)-maximal.

\[ \square \]

Lemma 8.13. Let \( \hat{w} \in D^{(m)}(n) \). The sketch \( \hat{w} \) is \( S \)-locally-maximal if and only if the tree \( \phi_m(\hat{w}) \) is in \( T_S \). In other words, \( \phi_m \) induces a bijection between \( S \)-locally-maximal regions of \( A_{[-m..m]}(N) \) and \( T_S \).

Proof. Let \( \hat{w} \in D^{(m)}(n) \) and let \( T = \phi_m(\hat{w}) \). For \( 0 \leq i < j \leq n \), and \( s \in [m] \), a \((i, j, s)\)-moves on \( \hat{w} \) is possible and gives an annotated \( m \)-sketch \( \hat{w}' \succ \hat{w} \) if and only if in \( \hat{w} \) the letter \( \alpha_j^{(s)} \) is immediately followed by \( \alpha_i^{(0)} \), and for all \( t \in [s+1..m] \), the letters \( \alpha_j^{(t)} \) and \( \alpha_i^{(t-s)} \) are consecutive. By definition of annotations, this holds if and only if the letter \( \alpha_j^{(s)} \) is immediately followed by \( \alpha_i^{(0)} \), and for all \( t \in [s+1..m] \), the letters \( \alpha_i^{(t)} \) is immediately followed by a \( \beta \)-letter. By Proposition 8.1, this holds if and only if in the tree \( T \) the node \( i \) is the \((s+1)st \) child of the node \( j \), and the right siblings of \( i \) are leaves (so that \( i = \text{cadet}(j) \) and \( \text{lsib}(i) = s \)).

Similarly, for \( 0 \leq i < j \leq n \), and \( s \in [-m..0] \), a \((i, j, s)\)-moves on \( \hat{w} \) is possible and gives an annotated \( m \)-sketch \( \hat{w}' \succ \hat{w} \) if and only if in \( \hat{w} \) the letter \( \alpha_i^{(-s)} \) is immediately followed by \( \alpha_j^{(0)} \) and for all \( t \in [-s+1..m] \), the letters \( \alpha_i^{(t)} \) is immediately followed by a \( \beta \)-letter. By Proposition 8.1, this holds if and only if in the tree \( T \) the node \( j \) is the \((s+1)st \) child of the node \( i \), and the right siblings of \( j \) are leaves (so that \( j = \text{cadet}(i) \) and \( \text{lsib}(j) = -s \)).

Thus \( \hat{w} \) is \( S \)-locally-maximal if and only if the tree \( T \) satisfies the following property for all \( 0 \leq i < j \leq n \) if \( i = \text{cadet}(j) \) then \( \text{lsib}(i) \in S_{i,j} \cup \{0\} \), and if \( j = \text{cadet}(i) \) then \( -\text{lsib}(j) \in S_{i,j} \). This holds if and only if \( T \) in \( T_S \).

We now complete the proof of Theorem 8.8. From Lemma 8.2, it is clear that for any tree \( T \in T^{(m)}(N) \), \( \Psi_S(T) \) is the region of \( A_S \) containing the region \( \Psi_m(T) \) of \( A_{[-m..m]}(N) \). Hence, by Lemma 8.13, the mapping \( \Psi_S \) is a surjection between the trees in \( T_S \) and the regions of \( A_S \) containing at least one \( S \)-locally-maximal region of \( A_{[-m..m]}(N) \). And since any \( S \)-maximal region is \( S \)-locally-maximal, Lemma 8.13 ensures that \( \Psi_S \) is a surjection between the trees in \( T_S \) and the regions of \( A_S \) (all this holds even if \( S \) is not transitive). Now assuming that \( S \) is transitive, Theorem 4.6 ensures that the regions of \( A_S \) are equinumerous to the trees in \( T_S \), so \( \Psi_S \) is actually a bijection.

9. Concluding remarks

We conclude with some additional links to the literature and some open questions.

9.1. Bijections for the Shi arrangement. We now explain how our bijection \( \Psi_{\{0,1\}} \) for the Shi arrangement \( A_{\{0,1\}}(n) \) relates to the existing bijections described in [7].
and [27] between regions of $A_{\{0,1\}}(n)$ and parking functions of size $n$. The correspondence is represented in Figure 14. Recall that a parking function of size $n$ is a $n$-tuple $(p_1, \ldots, p_n)$ of integers in $[0..n-1]$ such that for all $k \in [n]$, $k \leq |\{i \in [n] \mid p_i < k\}|$.

![Figure 14](image)

**Figure 14.** The bijection $\Psi_{\{0,1\}}$, the Athanasiadis-Linusson labeling and the Pak-Stanley labeling.

The first bijection discovered for the Shi arrangement is the so-called Pak-Stanley labeling of the regions described in [27] (and earlier in [30] Section 5] were Igor Pak is credited for suggesting the labeling in the case $m = 1$, without proof). This bijection associates to a region $\rho$ of $A_{\{0,1\}}(n)$ the parking function $(p_1, \ldots, p_n)$, where for all $i \in [n]$, $p_i = |\{k \in [i-1] \mid x_k < x_i\}| + |\{k \in [i+1..n] \mid x_k + 1 < x_i\}|$, where $(x_1, \ldots, x_n)$ is any point in the region $\rho$. This is represented in Figure 14(b). It follows directly from the definition of $\Psi_{\{0,1\}}$ that for any tree $T \in T_{\{0,1\}}(n)$, the Pak-Stanley labeling of the region $\rho = \Psi_{\{0,1\}}(T)$ is the parking function $\lambda_1(T) = (p_1, \ldots, p_n)$ given by $p_i = |\{k \in [i-1] \mid \text{node } k \prec_T \text{ node } i\}| + |\{k \in [i+1..n] \mid \text{right child of node } k \preceq_T \text{ node } i\}|$.

Another bijection for the Shi arrangement was established by Athanasiadis and Linusson in [7]. This bijection has two steps. The first step associates to each region $\rho$ of $A_{\{0,1\}}(n)$ a diagram $\delta(\rho)$. The second step associates to the diagram $\delta(\rho)$ a partition function that we call Athanasiadis-Linusson labeling of $\rho$. A reader familiar with [7] will have no difficulty seeing that the diagram $\delta(\rho)$ is closely related to the Shi-maximal 1-sketch that we associated to $\rho$ in Section 8.2. This induces a correspondence between our bijection and the Athanasiadis-Linusson labeling that we now state (the easy proof is omitted). For $T \in T_{\{0,1\}}(n)$, the Athanasiadis-Linusson labeling of the region $\rho = \Psi_{\{0,1\}}(T)$ is the parking function $\lambda_2(T) = (p_1, \ldots, p_n)$ obtained as follows. For all $i \in [n]$, we consider the path of vertices $v_1, v_2, \ldots , v_\ell$, where $v_1$ is the node $i$, $v_\ell$ is a leaf, and $v_{k+1}$ is the right child of $v_k$ for all $k \in [\ell - 1]$. Then, $p_i$ is the number of leaves greater than $v_\ell$ for the $\prec_T$ order. This is represented in Figure 14(c).
It is not very hard to see that the correspondence \( \lambda_2 \) is a bijection, and this is why the bijection in [7] can be considered a close relative of \( \Psi_{(0,1)} \). However, it is less clear why the correspondence \( \lambda_1 \) is a bijection.

9.2. Regions of the Linial arrangements and binary search trees. We now discuss the Linial arrangement \( A_{(1)}(n) \). Stanley had conjectured that the regions of \( A_{(1)}(n) \) were equinumerous to binary search trees with \( n \) nodes, that is, trees in \( T^{(1)}(n) \) satisfying the Condition (iii) of Figure 4. This fact was proved independently in [25] and [4]. In [25, 23] Postnikov and Stanley listed several combinatorial classes equinumerous to the regions of \( A_{(1)}(n) \), and some bijections between them. But, up to now, no bijection was known between these classes and the regions of \( A_{(1)}(n) \).

We remedy to this situation by giving bijections between regions of \( A_{(1)}(n) \), the set \( T_{(1)}(n) \) (which was not in the list), and the set \( B(n) \) of binary search trees with \( n \) nodes (which was in the list). The bijection between the regions of \( A_{(1)}(n) \) and \( T_{(1)}(n) \) was established in Section 8.2 (see also Figure 2). We now describe a recursive bijection \( \theta \), represented in Figure 15 between \( T_{(1)}(n) \) and \( B(n) \).

\[
\begin{align*}
\text{Figure 15. The recursive bijection } \theta \text{ from } T_{(1)}(n) \text{ to } B(n). \text{ In this example, exactly two of the trees } \theta(T_{j_1}), \theta(T_{j_2}), \theta(T_{j_3}) \text{ have no node, while the third tree is } \theta(T_p) \text{ for the only integer } p \in \{1, 2, 4, 6, 7\} \text{ such that } T_p \text{ has at least one node, and the root of } \theta(T_p) \text{ is less than } v_p.
\end{align*}
\]

For the tree \( \tau_0 \in T_{(1)}(0) \) made of one leaf, we define \( \theta(\tau_0) = \tau_0 \in B(0) \). We now consider \( n > 0 \) and suppose that \( \theta \) is a well defined bijection from \( T_{(1)}(k) \) to \( B(k) \) for all \( k < n \). By extension, we may assume that \( \theta \) is defined on all order-preserving relabeling of trees in \( T_{(1)}(k) \) for all \( k < n \) (with \( \theta \) preserving the set of labels). Let \( T \) be a tree in \( T_{(1)}(n) \), and let \( v_1 \) be its root. Let \( v_2, v_3, \ldots, v_{k+1} \) be defined by \( v_{i+1} = \text{cadet}(v_i) \) for all \( i \in [k] \), and the fact that both children of \( v_{k+1} \) are leaves. For \( i \in [k] \), let \( T_i \) be the subtree of \( T \) rooted at the child of \( v_i \) which is not \( v_{i+1} \); see Figure 15. We denote by \( I \) the subset of \( [k] \) such that either \( T_i \) is a reduced to a leaf which is the left child of \( v_i \), or the root of \( \theta(T_i) \) is a node which is greater than \( v_i \). Let \( i_1 < \cdots < i_a = k + 1 \) be the elements of \( I \cup \{k + 1\} \) and let \( k + 1 = j_1 > \cdots > j_b \) be the elements of \( [k + 1] \setminus I \). We then define \( \theta(T) \) as follows:

- \( v_{i_1} \) is the root,
- for all \( p \in [a - 1] \), the node \( v_{i_p} \) has right child \( v_{i_{p+1}} \) and left child the root of the subtree \( \theta(T_{i_p}) \),
for all \( p \in [b] \), the node \( v_{jp} \) has left child \( v_{jp+1} \) (or a leaf for \( p = b \)), and left child the root of the subtree \( \theta(T_{jp}) \).

It is easy to see that \( \theta(T) \) is in \( B(n) \) (since \( v_1 > v_2 > \cdots > v_k \)). It is also easy to see, by induction on \( n \), that \( \theta \) is a bijection (one of the useful observations to invert \( \theta \) is that \( \theta \) transforms subtrees \( T_j \) which are right leaves into subtrees which are right leaves).

9.3. **Open questions.** The braid arrangement is associated to the root system \( A_{n-1} \), in the sense that the hyperplane have the form \( \langle \alpha, x \rangle = 0 \) for the positive roots \( \alpha \) of \( A_{n-1} \). The (deformations of) arrangements corresponding to other root systems are known to share some of the properties of (deformations of) the braid arrangement (see e.g. [6, 25]). Thus, a natural question is whether the results of the present paper can be extended to this more general setting. Another direction for future research is to use the bijections presented here in order to obtain more refined counting formulas for the regions of deformed braid arrangements, by taking into account additional parameters of these regions (in the spirit of e.g. [27, 3]). We now state two open questions.

It was shown in Section 8.3, that when the tuple \( S \) is not transitive, the mapping \( \Psi_S \) still gives a surjection between the trees in \( T_S \) and the regions of \( A_S \). Indeed, \( \Psi_S \) gives a bijection between the subset of trees corresponding to \( S \)-maximal regions and the regions of \( A_S \).

**Question 9.1.** For a non-transitive tuple \( S \), is it possible to characterize a subset \( \tilde{T}_S \) of \( T_S \) in bijection with the regions of \( A_S \) via \( \Psi_S \)?

Let us consider for example the non-transitive set \( S = \{-2, 0, 2\} \). The set \( \tilde{T}_{\{-2,0,2\}}(n) \) contains all the trees in \( T^{(2)}(n) \) such that if the rightmost child of any node \( v \) is a leaf, then the middle child is also a leaf. However, because \( A_{\{-2,0,2\}}(n) \) is just a dilation of the Catalan arrangement \( A_{\{1,0,1\}}(n) \), we know that the regions of \( A_{\{-2,0,2\}}(n) \) are in bijection with the set \( T^{(1)}(n) \), or equivalently, the set \( \tilde{T}_{\{-2,0,2\}}(n) \) of trees in \( T_{\{-2,0,2\}}(n) \) such that the middle child of any node is a leaf. In general, one could hope to find the desired subset \( \tilde{T}_S \) of \( T_S \) either starting from the counting formulas in terms of boxed-trees (Theorem 4.2) and applying some sign-reversing involutions, or by using more direct bijective considerations.

A related problem is to find a more illuminating proof of our bijective results (Theorem 8.8). In the case of the Shi, semiorder, and Linial arrangement we gave a direct proof involving Lemma 8.5 showing that locally-maximal regions are maximal. The argument given there can actually be extended to the \( m \)-Shi, \( m \)-semiorder and \( m \)-Linial arrangements discussed in Section 2.3. However it is unclear whether such an approach would work in the general case (hence removing the need of using Theorem 4.6).

**Question 9.2.** Is there a direct, preferably geometric, proof that \( S \)-locally-maximal regions are \( S \)-maximal whenever \( S \) is transitive?

**References**


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