Computing the moments of the GOE bijectively

Olivier Bernardi (MIT)

Probability Seminar at MIT, February 2011
Computing the moments of the GOE bijectively

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2+3=5;
A combinatorial problem
Surfaces from a polygon

We consider the different ways of gluing the sides of a $2n$-gon in pairs.
Surfaces from a polygon

We consider the different ways of gluing the sides of a $2n$-gon in pairs.

The gluing of two sides can either be **orientable** (giving a cylinder) or **non-orientable** (giving a Möbius strip).

The surface obtained is orientable if and only if each gluing is orientable.
Surfaces from a polygon

We consider the different ways of gluing the sides of a $2n$-gon in pairs.

There are $(2n - 1)!! = (2n - 1)(2n - 3) \cdots 3$ ways of obtaining an orientable surface.
There are $2^n(2n - 1)!!$ ways of obtaining a general surface.
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**Question:** How many ways are there to obtain each surface (considered up to homeomorphism)?
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**Question:** How many ways are there to obtain each surface (considered up to homeomorphism)?

**Example:** The number of ways of getting the sphere is the Catalan number $\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$. 
Surfaces from a polygon

We consider the different ways of gluing the sides of a $2n$-gon in pairs. **Question:** How many ways are there to obtain a surface of type $t$?

By the Euler relation, the type of the surface is $t = n + 1 - \#\text{vertices}$. 
The Gaussian Orthogonal Ensemble
The GOE

Let $S_p$ be the set of real symmetric matrices of dimension $p \times p$.

We define a random variable $S$ in $S_p$ by choosing the entries $s_{i,j}$ for $i \leq j$ to be independent centered Gaussian variables with variance 2 if $i = j$ and variance 1 if $i < j$ (and then setting $s_{i,j} = s_{j,i}$ for $i > j$).
The GOE

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Hence, the distribution $\gamma$ of $S$ over $S_p$ has density $\kappa \exp(-\text{tr}(S^2)/4)$ with respect to the Lebesgue measure $dS := \prod_{i \leq j} ds_{i,j}$.

The GOE is the probability space $(S_p, \gamma)$. 
Eigenvalues of the GOE

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$ be the eigenvalues of $S$.

$S = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \\ \end{bmatrix}$

$S_{i,j} = S_{j,i}$

**Question:** What is the distribution of $\lambda := \lambda_U$, with $U$ uniform in $[p]$?
Eigenvalues of the GOE

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p$ be the eigenvalues of $S$. 

$$S = \begin{bmatrix} s_{i,j} = s_{j,i} \end{bmatrix}$$

**Question:** What is the distribution of $\lambda := \lambda_U$, with $U$ uniform in $[p]$?

The $n$th moment of $\lambda$ is 

$$\left\langle \frac{1}{p} \sum_{i=1}^{p} \lambda_i^n \right\rangle = \frac{1}{p} \langle \text{tr}(S^n) \rangle.$$ 

**Remark.** Odd moments are 0 by symmetry.
Computing the $2n$th moment using the Wick formula

We want the expectation of $\text{tr}(S^{2n}) = \sum_{i_1,i_2,...,i_{2n} \in [p]} s_{i_1,i_2}s_{i_2,i_3} \cdots s_{i_{2n},i_1}$.

Since the $s_{i,j}$ are Gaussian, the Wick formula gives

$$\langle s_{i_1,i_2}s_{i_2,i_3} \cdots s_{i_{2n},i_1} \rangle = \sum_{\pi \text{ matching on } [2n]} \prod_{\{k,l\} \in \pi} \langle s_{i_k,i_{k+1}}s_{i_l,i_{l+1}} \rangle.$$
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Contribution of matching \(\pi\)?
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Contribution of matching $\pi$?

Hint: $\langle s_{i_k, i_{k+1}} s_{i_l, i_{l+1}} \rangle = 0$ except if $(i_k, i_{k+1}) = (i_l, i_{l+1})$ or $(i_{l+1}, i_l)$. 

[Diagram of a graph with nodes labeled $i_1, i_2, \ldots, i_{2n}$]
Computing the $2n$th moment using the Wick formula

$$
\langle \text{tr}(S^{2n}) \rangle_p = \sum_{\pi \text{ matching on } [2n]} \sum_{i_1...i_{2n}\in[p]} \prod_{\{k,l\}\in\pi} \langle s_{i_k,i_{k+1}} s_{i_l,i_{l+1}} \rangle
$$

$$
= \sum_{\pi \text{ matching on } [2n]} \sum_{i_1...i_{2n}\in[p]} \prod_{\{k,l\}\in\epsilon^{-1}(N)} 1_{i_k=i_l, i_{k+1}=i_{l+1}}
$$

$$
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$$
= \sum_{\text{gluing of } 2n\text{-gon}} p^{\#\text{vertices}}.
$$
Computing the $2n$th moment using the Wick formula

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$$= \sum_{\pi \text{ matching on } [2n]} \sum_{i_1 \ldots i_{2n} \in [p]} \prod_{\{k,l\} \in \epsilon^{-1}(N)} 1_{i_k = i_l, i_{k+1} = i_{l+1}} \times \prod_{\{k,l\} \in \epsilon^{-1}(O)} 1_{i_k = i_{l+1}, i_{k+1} = i_l}$$

$$= \sum_{\text{gluing of } 2n\text{-gon}} p^{\#\text{vertices}}.$$

**Summary:** The $2n$th moment of $\lambda$ in GOE($p$) is

$$\frac{1}{p} \sum_{\text{gluings of } 2n\text{-gon}} p^{\#\text{vertices}}.$$
Computing the $2n$th moment using the Wick formula

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$$\times \prod_{\{k,l\} \in \epsilon^{-1}(O)} 1_{i_k = i_{l+1}, i_{k+1} = i_l}$$

$$= \sum_{\text{gluing of } 2n\text{-gon}} p^{\#\text{vertices}}.$$

**Summary:** The $2n$th moment of $\lambda$ in GOE$(p)$ is $\frac{1}{p} \sum p^{\#\text{vertices}}$.

**Remark.** When $p \to \infty$ the $2n$th moment of $\frac{\lambda}{\sqrt{p}}$ which is $\sum p^{\#\text{vertices} - n - 1}$ tends to $\text{Cat}(n)$. $\Rightarrow$ Semi-circle law.
Back to the polygon
Surfaces from a polygon: state of the art

**Question:** How many ways of gluing $2n$-gon into a surface of type $t$?
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\[ \eta_v(n) \]

<table>
<thead>
<tr>
<th>$n$ = #edges</th>
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<th>374</th>
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<tr>
<td>$v$ = #vertices</td>
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Surfaces from a polygon: state of the art
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**Question:** How many ways of gluing $2n$-gon into a surface of type $t$?

\[ \sum_v \epsilon_v(n)p^v \text{ and recurrence formula for } \epsilon_v(n). \]

**Results for orientable surfaces:**

\[
\begin{array}{cccccc}
1 & 1 & 5 & 52 & 22 & 5 \\
1 & 5 & 41 & 509 & 690 & 374 & 93 & 14
\end{array}
\]

[Harer, Zagier 86]
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$$v = \#\text{vertices}$$

$$n = \#\text{edges}$$

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**Results for orientable surfaces:**

Formula for $\sum_v \epsilon_v(n)p^v$ and recurrence formula for $\epsilon_v(n)$.

$$\sum_v \epsilon_v(n)p^v = \sum_{q=1}^{p} \binom{p}{q}2^{q-1}\binom{n}{q-1}(2n-1)!!$$

$$(n + 1)\epsilon_v(n) = (4n - 2)\epsilon_{v-1}(n - 1) + (n - 1)(2n - 1)(2n - 3)\epsilon_v(n - 2).$$

**Related work:** Jackson, Adrianov, Zagier, Mehta, Haagerup-Thorbjornsen, Kerov, Ledoux, Lass, Goulden-Nica, Schaeffer-Vassilieva, Morales-Vassilieva, Chapuy...
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[Ledoux 09] Recurrence formula for $\eta_v(n)$.
[B., Chapuy 10] Asymptotic $\eta^t(n) \sim_{n \to \infty} c_t n^{3(t-1)/2} 4^n$,

where $c_t = \left\{ \begin{array}{ll}
\frac{2^{t-2}}{\sqrt{6}^{t-1}(t-1)!!} & \text{if } t \text{ odd}, \\
\frac{3 \cdot 2^{t-2}}{\sqrt{\pi} \sqrt{6}^t(t-1)!!} \sum_{i=1}^{t/2-1} \binom{2i}{i} 16^{-i} & \text{if } t \text{ even.} 
\end{array} \right.$
Results

**Theorem [B.]:** The number of ways of gluings a $2n$-gon with vertices colored using every color in $[q]$ is

$$\sum_{r=1}^{\frac{n-q+2}{2}} \frac{q!r!}{2^{r-1}} P_{q,r} \left(\frac{2n}{2q + 2r - 4}\right)(2n - 2q - 2r + 1)!!$$

where $P_{q,r}$ is the number of **planar maps** with $q$ vertices and $r$ faces.

That is, $P_{q,r}$ is the coefficient of $x^q y^r$ in the series $P(x, y)$ defined by:

$$27P^4 - (36x + 36y - 1)P^3 + (24x^2 y + 24xy^2 - 16x^3 - 16y^3 + 8x^2 + 8y^2 + 46xy - x - y)P^2 + xy(16x^2 + 16y^2 - 64xy - 8x - 8y + 1)P - x^2y^2(16x^2 + 16y^2 - 32xy - 8x - 8y + 1) = 0.$$
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Theorem [B.]: The number of ways of gluings a $2n$-gon with vertices colored using every color in $[q]$ is

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where $P_{q,r}$ is the number of planar maps with $q$ vertices and $r$ faces.

Theorem [Harer-Zagier 86]: The number of orientable ways of gluings a $2n$-gon with vertices colored using every color in $[q]$ is

$$2^{q-1} \binom{n}{q-1}(2n - 1)!!$$
Results

**Theorem [B.]:** The number of ways of gluings a \(2n\)-gon with vertices colored using every color in \([q]\) is

\[
\sum_{r=1}^{n-q+2} \frac{q!r!}{2r-1} P_{q,r} \left( \frac{2n}{2q + 2r - 4} \right)(2n - 2q - 2r + 1)!!
\]

where \(P_{q,r}\) is the number of planar maps with \(q\) vertices and \(r\) faces.

\(\Rightarrow\) The \(2n\)th moment of \(\lambda\) in the GOE(p) = \(\frac{1}{p} \sum_{\text{gluings of } 2n\text{-gon}} \text{gluings of } p\#\text{vertices} \)

\[
= \frac{1}{p} \#\text{gluings of } 2n\text{-gon with vertices colored in } [p],
\]

\[
= \frac{1}{p} \sum_{q=1}^{p} \binom{p}{q} \sum_{r=1}^{n-q+2} \frac{q!r!}{2r-1} P_{q,r} \left( \frac{2n}{2q + 2r - 4} \right)(2n - 2q - 2r + 1)!!
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where $P_{q,r}$ is the number of planar maps with $q$ vertices and $r$ faces.

⇒ The $2n$th moment of $\lambda$ in the GOE($p$) = $\frac{1}{p} \sum$ gluings of $p$-#vertices

$$= \frac{1}{p} \# \text{gluings of } 2n\text{-gon with vertices colored in } [p],$$

$$= \frac{1}{p} \sum_{q=1}^{p} \binom{p}{q} \sum_{r=1}^{n-q+2} \frac{q!r!}{2r-1} P_{q,r} \left(\frac{2n}{2q + 2r - 4}\right)(2n - 2q - 2r + 1)!!$$

**Theorem [Goulden, Jackson 97]** This number is

$$n! \sum_{k=0}^{n} 2^{2n-k} \sum_{r=0}^{n} \binom{n}{n-r} \binom{k+r-1}{k} \binom{p-1}{r} + (2n-1)!! \sum_{q=1}^{p-1} 2^{q-1} \binom{p-1}{q} \binom{n}{q-1}.$$
Results

**Corollary [recovering Ledoux 09]:** The number $\eta_v(n)$ of ways of gluings a $2n$-gon into a surfaces of type $n + 1 − v$ satisfies:

$$(n + 1) \eta_v(n) = (4n - 1) (2 \eta_{v-1}(n-1) - \eta_v(n-1))$$
$$+ (2n - 3) [(10n^2 - 9n) \eta_v(n-2) + 8 \eta_{v-1}(n-2) - 8 \eta_{v-2}(n-2)]$$
$$+ 5(2n - 3)(2n - 4)(2n - 5)(\eta_v(n-3) - 2 \eta_{v-1}(n-3))$$
$$- 2(2n - 3)(2n - 4)(2n - 5)(2n - 6)(2n - 7) \eta_v(n-4).$$
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$$+ 5 (2n - 3)(2n - 4)(2n - 5) (\eta_v(n-3) - 2 \eta_{v-1}(n-3))$$

$$- 2(2n - 3)(2n - 4)(2n - 5)(2n - 6)(2n - 7) \eta_v(n-4).$$

**[Harer, Zagier 86]:** The number $\epsilon_v(n)$ of **orientable ways** of gluings a $2n$-gon into a surfaces of type $n + 1 - v$ satisfies:

$$(n + 1) \epsilon_v(n) = (4n - 2) \epsilon_{v-1}(n - 1) + (n - 1)(2n - 1)(2n - 3) \epsilon_v(n - 2).$$
Crash course on maps

Definition: A map is a graph embedded in a surface obtained by gluing together polygons (i.e. the faces are simply connected).
Crash course on maps

Definition: A map is a graph embedded in a surface obtained by gluing together polygons (i.e. the faces are simply connected). Maps are considered up to homeomorphism.
Crash course on maps

**Definition:** A map is a graph embedded in a surface obtained by gluing together polygons (i.e. the faces are simply connected). Maps are considered *up to homeomorphism*.

A **planar map** is a map on the sphere.
A **unicellular map** is a map with 1 face.
A **rooted map** is a map with a marked edge-direction+side.

**Example:** the rooted unicellular planar maps are the "rooted plane trees".
Crash course on maps

Definition: A map is a graph embedded in a surface obtained by gluing together polygons (i.e. the faces are simply connected). Maps are considered up to homeomorphism.

Embedding Thorem: Rooted maps on orientable surfaces are in bijection with graph+rooted rotation system, that is,
- a total order on the half-edges around a “root-vertex”,
- a cyclic order on the half-edges around the other vertices.
Bijections for $q$-colored unicellular maps

A **tree-rooted map** is a map on an orientable surface with a marked spanning tree. 
A **planar-rooted map** is a map on an orientable surface with a marked planar connected spanning submap.

**Remark.** The number of tree-rooted maps with $q$ vertices and $n$ edges is

$$\text{Cat}(q-1) \binom{2n}{2q}(2n - 2q + 1)!! = \frac{2^{q-1}}{q!} \binom{n}{q-1}(2n - 1)!!$$

The number of planar-rooted maps with $q$ vertices, $r$ faces, and $n$ edges is

$$P_{q,r} \binom{2n}{2q + 2r - 4}(2n - 2q - 2r + 1)!!$$
Bijections for $q$-colored unicellular maps

**Thm [B.]** There is a bijection between rooted unicellular map colored using every color in $[q]$ with $n$ edges on orientable surfaces and tree-rooted maps with $q$ labelled vertices and $n$ edges.

Moreover, number of edges between colors $i$ and $j \rightarrow$ number of edges between vertices $i$ and $j$.

![Bijection Diagram]

**Corollary [Harer-Zagier]** The number of $[q]$-colored rooted unicellular maps with $n$ edges on orientable surfaces is: $2^{q-1} \binom{n}{q-1} (2n-1)!!$

**Previous combinatorial proofs:** [Lass 01], [Goulden,Nica 05].

**Bipartite case:** [Schaeffer,Vassilieva 08], [Morales,Vassilieva 09].
Bijections for $q$-colored unicellular maps

**Thm [B.]** There is a $q!r!2^{1-r}$-to-1 correspondence $\Phi$ between rooted unicellular maps on **general surfaces** colored using every color in $[q]$ with $n$ edges and **planar-rooted maps** with $q$ vertices, $r$ faces and $n$ edges.

Moreover, number of edges incident to color $i \mapsto$ degree of vertex $i$.

**Corollary:** The number of $[q]$-colored rooted unicellular maps with $n$ edges on general surfaces is:

$$\sum_{r=1}^{n-q+2} \frac{q!r!}{2^{r-1}} P_{q,r} \left(\binom{2n}{2q + 2r - 4}(2n - 2q - 2r + 1)!!\right)$$

where $P_{q,r}$ is the number of **planar maps** with $q$ vertices and $r$ faces.
Sketch of the proof
Idea 1: Colored unicellular map $\leftrightarrow$ bi-Eulerian tour

**Def.** Let $G = (V, E)$ be a graph. A **bi-Eulerian tour** is a walk starting and ending at the same vertex and using every edge twice.
Idea 1: Colored unicellular map ↔ bi-Eulerian tour

Def. Let $G = (V, E)$ be a graph. A bi-Eulerian tour is a walk starting and ending at the same vertex and using every edge twice.

Lemma [adapting Lass 01]. [$q$]-colored unicellular maps are in bijection with pairs $(G, E)$, where $G$ is a graph with vertex set $[q]$ and $E$ is a bi-Eulerian tour.

Moreover, the map is on an orientable surface if and only if no edge is used twice in the same direction.
Idea 2: BEST Theorem

BEST Theorem. Let $\vec{G} = (V, A)$ be a directed graph such that every vertex has as many ingoing and outgoing arcs. The Eulerian tours of $\vec{G}$ starting and ending at $v$ are in bijection with pairs $(T, O)$ made of a spanning tree $T$ oriented toward $v +$ ordering $O$ around each vertex of the outgoing arc not in $T$.
Idea 2: BEST Theorem

**BEST Theorem.** Let $\vec{G} = (V, A)$ be a directed graph such that every vertex has as many ingoing and outgoing arcs. The Eulerian tours of $\vec{G}$ starting and ending at $v$ are in bijection with pairs $(T, O)$ made of a spanning tree $T$ oriented toward $v$ + ordering $O$ around each vertex of the outgoing arc not in $T$. 
**Idea 2: BEST Theorem + embedding Theorem**

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**Corollary 1.** Let $G$ be a graph. The bi-Eulerian tours of $G$ which never take an edge twice in the same direction are in bijection with pairs $(T, R)$ made of a spanning tree + rooted rotation system. $\leftrightarrow$ **tree-rooted maps** having underlying graph $G$. 
Idea 2: BEST Theorem + embedding Theorem

⇒ **Bijection** between rooted unicellular maps colored using every color in $[q]$ with $n$ edges on **orientable surfaces** and tree-rooted maps with $q$ labelled vertices and $n$ edges.
Idea 2: BEST Theorem + embedding Theorem

A **bi-oriented tree-rooted map** is a rooted map on orientable surface + spanning tree + partial orientation such that
- indegree = outdegree for every vertex
- the oriented edges in the spanning tree toward parent.
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**Corollary 2.** Let $G$ be a graph. There is a $1$-to-$2^r$ correspondence between the bi-Eulerian tours of $G$ and the bi-oriented tree-rooted map on $G$ having $r$ oriented edges outside of the spanning tree.
Idea 3: cutting and pasting

Let $B$ be a bi-oriented tree-rooted map. The planar-rooted map $P = \Psi(B)$ is obtained by cutting the oriented external edges in their middle and regluing them according to the parenthesis system they form around the tree (and then forgetting the tree + orientation).

\[
(r-1)! \text{-to-1} \quad \text{versus} \quad r \text{-to-1}
\]

**Theorem:** The mapping $\Psi$ is $r!$-to-$1$ between bi-oriented tree-rooted maps with $r - 1$ oriented edges outside of the spanning tree and planar-rooted map with $r$ sub-faces.
Idea 3: cutting and pasting

$\Rightarrow q!r!2^{r-1}$-to-1 correspondence between rooted unicellular maps colored using every color in $[q]$ with $n$ edges on general surfaces and planar-rooted maps with $q$ vertices, $r$ faces, and $n$ edges.
Recurrence relation
Toward the recurrence relation

We want to prove the recurrence of Ledoux:

\[(n + 1) \eta_v(n) = (4n - 1) (2 \eta_{v-1}(n-1) - \eta_v(n-1)) + (2n - 3) \left( (10n^2 - 9n) \eta_v(n-2) + 8 \eta_{v-1}(n-2) - 8 \eta_{v-2}(n-2) \right) + 5(2n - 3)(2n - 4)(2n - 5) (\eta_v(n-3) - 2 \eta_{v-1}(n-3)) - 2(2n - 3)(2n - 4)(2n - 5)(2n - 6)(2n - 7) \eta_v(n-4). \]
Toward the recurrence relation

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\[+ (2n - 3) ((10n^2 - 9n) \eta_v(n-2) + 8 \eta_{v-1}(n-2) - 8 \eta_{v-2}(n-2))
\]  
\[+ 5(2n - 3)(2n - 4)(2n - 5) (\eta_v(n-3) - 2 \eta_{v-1}(n-3))
\]  
\[- 2(2n - 3)(2n - 4)(2n - 5)(2n - 6)(2n - 7) \eta_v(n-4).
\]

Let \( V_n(x) = \frac{1}{(2n)!} \sum_{v=1}^{n+1} \eta_v(n)x^n \). The recurrence is equivalent to

\[\begin{aligned}
- (2n)(2n - 1)(2n - 2)(n + 1)V_n(x) + (4n - 1)(2n - 2)(2x - 1)V_{n-1}(x)
+ (10n^2 - 9n + 8x - 8x^2)V_{n-2}(x) + 5(1 - 2x)V_{n-3}(x) - 2V_{n-4}(x) &= 0.
\end{aligned}\]
Toward the recurrence relation

We want to prove the recurrence of Ledoux:

\[(n + 1) \eta_v(n) = (4n - 1)(2\eta_{v-1}(n-1) - \eta_v(n-1)) \]
\[+ (2n - 3)((10n^2 - 9n) \eta_v(n-2) + 8 \eta_{v-1}(n - 2) - 8 \eta_{v-2}(n - 2)) \]
\[+ 5(2n - 3)(2n - 4)(2n - 5)(\eta_v(n-3) - 2 \eta_{v-1}(n-3)) \]
\[\quad - 2(2n - 3)(2n - 4)(2n - 5)(2n - 6)(2n - 7) \eta_v(n-4).\]

Let \( V_n(x) = \frac{1}{(2n)!} \sum_{v=1}^{n+1} \eta_v(n)x^n. \) The recurrence is equivalent to

\[-(2n)(2n-1)(2n-2)(n+1)V_n(x) + (4n-1)(2n-2)(2x-1)V_{n-1}(x) \]
\[+ (10n^2 - 9n + 8x - 8x^2)V_{n-2}(x) + 5(1-2x)V_{n-3}(x) - 2V_{n-4}(x) = 0.\]

\[V_n(x) = \frac{1}{(2n)} \sum_{q=1}^{n+1} \binom{x}{q} U_n(q), \] where \( U_n(q) \) is the number of unicellular maps with \( n \) edges colored using every color in \([q]\).

\(\Rightarrow\) my computer can translate (*) into a linear differential equation for the series \( U(t, w) = \sum_{n,q} U_n(q) (2n)! t^n w^q. \)
Toward the recurrence relation

We know \[ U_n(q) = \sum_{r=1}^{n-q+2} \frac{q!r!}{2r-1} P_{q,r} \left( \frac{2n}{2q + 2r - 4} \right) (2n - 2q - 2r + 1)!! \].

\[ \Rightarrow U(t, w) = \frac{1}{2} \exp \left( \frac{t}{2} \right) Q \left( \frac{t}{2}, 2w \right), \]

where \[ Q(t, w) = \sum_{q,r > 0} \frac{q!r! P_{q,r}}{(2q + 2r - 4)!} t^{q+r-2} w^q. \]
Toward the recurrence relation

We know \( U_n(q) = \sum_{r=1}^{n-q+2} \frac{q!r!}{(2r-1)!} P_{q,r} \left( \frac{2n}{2q + 2r - 4} \right) (2n - 2q - 2r + 1)!! \).

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Equation (*) is equivalent to:

\[
(6tw + 4w^2 - 36t - 7w - 6)Q(t, w) - (12t^2 + 8tw + 7w - 25) \frac{\partial}{\partial t} Q(t, w) \\
+ w(w+2)(8w+6t-7) \frac{\partial^2}{\partial w \partial t} Q(t, w) + (2t^2 - 4tw + 37t + 9) \frac{\partial^2}{\partial t^2} Q(t, w) \\
- w(w+2)(8t+7) \frac{\partial^2}{\partial w^2} Q(t, w) + 2w^2(w+2)^2 \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial w^2} Q(t, w) \\
+ t(8t+11) \frac{\partial^3}{\partial t^3} Q(t, w) - 4wt(w+2) \frac{\partial^3}{\partial w \partial t^2} Q(t, w) + 2t^2 \frac{\partial^4}{\partial t^4} Q(t, w) = 0.
\]
Toward the recurrence relation

We know

\[ U_n(q) = \sum_{r=1}^{n-q+2} \frac{q!r!}{2r-1} P_{q,r} \left( \frac{2n}{2q + 2r - 4} \right) (2n - 2q - 2r + 1)!! \]  

\[ \Rightarrow U(t, w) = \frac{1}{2} \exp \left( \frac{t}{2} \right) Q \left( \frac{t}{2}, 2w \right), \]

where \( Q(t, w) = \sum_{q,r>0} \frac{q!r! P_{q,r}}{(2q + 2r - 4)!} t^{q+r-2} w^q. \)

Equation (*) is equivalent to:

\[ 0 = (4y - 2x - 1) \left( 72P(x, y) - 72x \frac{\partial}{\partial x} P(x, y) - (72y - 2) \frac{\partial}{\partial y} P(x, y) \right) \]
\[ -72x^2 \frac{\partial^2}{\partial x^2} P(x, y) + (8x^2 - 6x - 24xy - 56y^2 + 10y + 1) \frac{\partial^2}{\partial y^2} P(x, y) \]
\[ \Rightarrow 2x (4x - 80y - 1) \frac{\partial^2}{\partial y \partial x} P(x, y) \]
\[ + (4x - 8y - 1) \left( y (4x + 48y^2 - 8y - 1) \frac{\partial^3}{\partial y^3} P(x, y) + 48x^3 \frac{\partial^3}{\partial x^3} P(x, y) \right) \]
\[ + (4x - 8y - 1) \left( 144x^2 y \frac{\partial^3}{\partial y \partial x^2} P(x, y) + x (4x - 8y - 1 + 144y^2) \frac{\partial^3}{\partial y^2 \partial x} P(x, y) \right) \]

where \( P(x, y) = \sum_{q,r>0} P_{q,r} x^q y^r. \)
Toward the recurrence relation

The differential equation (**) for the series \( P(x, y) \) of planar maps can be checked using the algebraic equation:

\[
27P^4 - (36x + 36y - 1)P^3 \\
+ (24x^2y + 24xy^2 - 16x^3 - 16y^3 + 8x^2 + 8y^2 + 46xy - x - y)P^2 \\
+ xy(16x^2 + 16y^2 - 64xy - 8x - 8y + 1)P \\
- x^2y^2(16x^2 + 16y^2 - 32xy - 8x - 8y + 1) = 0.
\]

(by expressing the partial derivatives of \( P \) as polynomials in \( P \)).  \( \square \)
Thanks.