Random colored lattices

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Joint work with Mireille Bousquet-Mélou (CNRS)

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Random lattices, Random surfaces
Maps

A **map** is a way of gluing polygons in order to form a connected surface.
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A planar map is a map on the sphere.

\[\text{maps} \]

\[\text{planar maps} \]

\[=\]
A random lattice?

Define a random lattice by choosing uniformly among all ways of gluing $n = 1000$ squares to form a planar map.
A random lattice?

Define a **random lattice** by choosing uniformly among all ways of gluing $n = 1000$ squares to form a planar map.

**Theorem [Tutte 62]** There are \[ \frac{2 \cdot 3^n}{(n + 1)(n + 2)} \binom{2n}{n} \] rooted planar quadrangulations with $n$ squares.
A random lattice?

Define a random lattice by choosing uniformly among all ways of gluing $n = 1000$ squares to form a planar map.

What kind of random object is it?
- A random graph.
- A random metric space: $\langle V, d \rangle$. 
A random lattice?

Define a random lattice by choosing uniformly among all ways of gluing $n = 1000$ squares to form a planar map.

Properties of random lattices?
- What is the typical distance between two points?
- Are there small cycles separating two big pieces of the map?
A random lattice?

Define a **random lattice** by choosing uniformly among all ways of gluing $n = 1000$ squares to form a planar map.

Properties of random lattices?

- **What is the typical distance** between two points?

**Theorem [Chassaing, Schaeffer 03]** The distances scale as $C_n n^{1/4}$, where $C_n$ converges in law toward the law of the ”Integrated Super-Brownian Excursion” (up to a constant factor).
A random lattice?

Define a **random lattice** by choosing uniformly among all ways of gluing $n = 1000$ squares to form a planar map.

Properties of random lattices?
- **What is the typical distance** between two points?
  Theorem [Chassaing, Schaeffer 03] The distances scale as $C_n n^{1/4}$.

- **Are there small cycles** separating two big pieces of the map?
  Theorem [LeGall 07, Miermont 09] No: if $\epsilon > 0$ and $f(n) \to 0$ then
  $$\mathbb{P}(\text{exists cycle of length } < f(n) n^{1/4} \text{ separating two pieces } > \epsilon n) \to 0$$
How are such results proved?

**Bijection.** [Cori Vauquelin 81, Schaeffer 98]

Rooted pointed quadrangulation with $n$ faces.

Rooted plane trees with $n$ edges + vertex labels changing by $-1$, $0$, $1$ along edges and such that $\text{min} = 1$. 
How are such results proved?

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Rooted pointed quadrangulation with \( n \) faces.

\[
(n + 2) \cdot \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n} = 2\text{-to-1}
\]

Rooted plane trees with \( n \) edges + vertex labels changing by \(-1, 0, 1\) along edges and such that \( \min = 1 \).

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3^n \cdot \frac{1}{(n + 1)} \binom{2n}{n} = 2\text{-to-1}
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Vertex at distance $d$ from pointed vertex

Vertex labeled $d$
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Vertex at distance \( d \) from pointed vertex

Vertex labeled \( d \)
Random surfaces?

Let $Q_n$ be the metric space $(V, \frac{d}{n^{1/4}})$ corresponding to a uniformly random quadrangulations with $n$ faces.
Random surfaces?

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**Theorem.** [Le Gall 2007, Miermont/Le Gall 2012] The sequence $Q_n$ converges in distribution (in the Gromov Hausdorff topology) toward a random metric space, which
- is homeomorphic to the sphere
- has Hausdorff dimension 4.

**Related work.** Bouttier, Di Francesco, Chassaing, Guitter, Le Gall, Marckert, Miermont, Paulin, Schaeffer, Weill . . .
Random surfaces?

The **Brownian map** is described in terms of
- Continuum Random Tree (≡ limit of discrete trees)
- Gaussian labels (≡ limit of discrete labels)
Random surfaces?

The **Brownian map** is described in terms of
- Continuum Random Tree (= limit of discrete trees)
- Gaussian labels (= limit of discrete labels)

The Brownian map is **universal**: it is also the limit of uniformly random triangulations, etc.
Right notion of random surface?

**Motivation:** 2D quantum gravity.

“There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. *In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitudes are given by the sums over all possible surfaces with fixed boundary.*”

(A.M. Polyakov, Moscow, 1981.)
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Two candidates for random surfaces:
- Discrete surfaces $\rightsquigarrow$ Brownian map
- Gaussian Free Field
A glimpse at the Gaussian Free Field

Discrete version: Let $h(x, y)$ be a function on $\{1, 2, \ldots, n\}^2$ chosen with probability density proportional to

$$\exp\left(-\frac{1}{2} \sum_{e=(u,v)} (h(u) - h(v))^2\right) dh.$$
A glimpse at the Gaussian Free Field

Discrete version: Let \( h(x, y) \) be a function on \( \{1, 2, \ldots, n\}^2 \) chosen with probability density proportional to

\[
\exp\left( -\frac{1}{2} \sum_{e=(u,v)} (h(u) - h(v))^2 \right) \, dh.
\]

Use \( h \) to define a (random) ”density of area” on the square:

\[
\text{Area}(v) := \exp(\gamma h),
\]

and the (random) length of paths:

\[
\text{Length}(P) := \sum_{v \in P} \exp(\gamma h/2),
\]

where \( \gamma \in [0, 2) \).
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Continuous version: One would like to choose \( h \) on \([0, 1]^2\) with a probability density proportional to

\[
\exp\left(-\frac{1}{2} \int_{[0,1]^2} ||\nabla h||^2\right) dh,
\]

but does not make sense...
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but does not make sense...

Solution: Take an orthonormal basis of functions $\{f_i\}_{i>0}$, and define $h := \sum_i \alpha_i f_i$ with $\alpha_i$ Gaussian.

The GFF $h$ is **not a function but a distribution**: it is not defined at points, but the average in any region is well defined.
Some big questions

Question 1.
Relation between Brownian map and Gaussian Free Field?
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Parameter $\gamma$ of GFF in terms of random maps?
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Relation between Brownian map and Gaussian Free Field?

Question 2.
Parameter $\gamma$ of GFF in terms of random maps?

Prediction from physics: The parameter $\gamma$ of GFF is associated to the central charge $c$ by

$$\gamma = \frac{\sqrt{25 - c} - \sqrt{1 - c}}{\sqrt{6}}.$$  

and certain value of $c$ can be achieved by adding a statistical mechanics model on the map.
Example: No model is $c = 0$ and Ising model is $c = 1/2$. 
Maps chosen according to a statistical mechanics model.

**Idea:** Instead of choosing a planar map of size $n$ uniformly at random, let us choose it according to a statistical mechanics model.

**Example:** *Spanning-tree model.* Choose maps with probability proportional to number of spanning trees.
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**Idea:** Instead of choosing a planar map of size $n$ uniformly at random, let us choose it according to a statistical mechanics model.

**Example:** *Spanning-tree model.* Choose maps with probability proportional to number of spanning trees.

**Theorem [Mullin 67, Bernardi 07].** There are $\text{Cat}_n \text{Cat}_{n+1}$ rooted maps + spanning tree, where $\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}$. 

![Diagram of a map with red spanning tree]
Maps chosen according to a statistical mechanics model.

Question. Is the "spanning-tree model" map really different?
Maps chosen according to a statistical mechanics model.

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**Simulation [Bernardi, Chapuy]** The Hausdorff dimension of the spanning-tree model map is $D_H \approx 3.5(5)$.

**Conj. [Watabiki 93]**

$$D_H = 2 \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}} \approx 3.56$$

where $c = -2$ is central charge.
Maps chosen according to a statistical mechanics model.

**Question.** Is the ”spanning-tree model” map really different?

**Easier information:** **asymptotic number of maps.**

Number of (rooted) maps with \( n \) edges is \( M_n \sim k n^{-5/2} R^n \).

Number of maps+spanning trees: \( T_n \sim k' n^{-3} R'^n \).
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Number of maps+spanning trees: $T_n \sim k' n^{-3} R'^n$

**Theorem [Bender, Canfield 97+ Bernardi, Fusy 12]**
The exponent $-5/2$ is universal: same thing for any family of maps defined by face degree constraints $+$ girth constraints.
The exponent is $-3$ also universal.
Maps chosen according to a statistical mechanics model.

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Number of maps+spanning trees: $T_n \sim k' n^{-3} R'^n$

**Prediction from physics:**

$$\alpha = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12},$$

where $c$ is the central charge ($c = 0$ for maps without model, $c = -2$ for spanning-tree model).
Maps chosen according to a statistical mechanics model.

**Question.** Is the "spanning-tree model" map *really different*?

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**KPZ formula [Knizhnik, Polyakov, Zamolodchikov 88]:** Conjecture relating critical exponents in regular lattices Vs random lattices.
Asymptotic number of maps and geometry?

Suppose number of objects of size $n$ is $M_n \sim k n^{-\alpha} R^n$.

Sample a pair of objects of total size $n$. How small is the smallest one?
Asymptotic number of maps and geometry?

Suppose number of objects of size $n$ is $M_n \sim k n^{-\alpha} R^n$.

Sample a pair of objects of total size $n$. How small is the smallest one?

Size of smallest one is $s$ with probability proportional to $\approx \frac{k^2 R^n}{(s(n - s))^\alpha}$.

Thus, $\alpha$ big makes $\frac{s}{n}$ small.
Potts model on maps
Potts model  (parameters $q$, $\nu$)

**Configurations** of the $q$-Potts model on a graph $G$ are $q$-colorings. Their **weight** is $\nu \# \text{monochromatic edges}$.
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The **partition function of the Potts model** on the graph $G$ is

$$P_G(q, \nu) = \sum_{q\text{-coloring}} \nu\#\text{monochromatic edges}.$$
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Each coloring appears with **probability** $\frac{\nu \#\text{monochromatic edges}}{P_G(q, \nu)}$. 
Potts model

\[ P_G(q, \nu) = \sum_{q-\text{coloring}} \nu^{\#\text{monochromatic edges}}. \]

- The case \( q = 2 \) is the Ising model (without external field).
- The special case \( P_G(q, 0) \) counts the proper \( q \)-colorings of \( G \).
The Potts model is given by

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- The special case \( P_G(q, 0) \) counts the proper \( q \)-colorings of \( G \).
- \( P_G(q, \nu) \) satisfies a recurrence relation:

\[ P_G(q, \nu) = P_{G \setminus e}(q, \nu) + (\nu - 1) P_{G/e}(q, \nu). \]
Potts model

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- The case \( q = 2 \) is the **Ising model** (without external field).
- The special case \( P_G(q, 0) \) counts the **proper** \( q \)-colorings of \( G \).
- \( P_G(q, \nu) \) satisfies a **recurrence relation**:

\[ P_G(q, \nu) = P_G\backslash e(q, \nu) + (\nu - 1) P_G/ e(q, \nu). \]

- \( P_G(q, \nu) \) is a polynomial in the variables \( q, \nu \)
  It is equivalent to the **Tutte polynomial**
  (a.k.a. partition function of the **FK-Cluster model**).
The Potts model on planar maps

The **partition function** of the (annealed) **Potts model on maps** is

\[ G(q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu) z^{\# \text{edges}}. \]

This is a series in \( z \) with coefficients polynomial in \( q, \nu \).
The Potts model on planar maps

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This is a series in $z$ with coefficients polynomial in $q, \nu$.

Each map appears with **probability** $\frac{P_M(q, \nu) z^{\# \text{edges}}}{G(q, \nu, z)}$. 
The Potts model on planar maps

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**Goals:**
- Characterize the partition function $G(q, \nu, z)$. 

The Potts model on planar maps

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Goals:

- Characterize the partition function \( G(q, \nu, z) \).
- Study the phase transitions.

Example for \( q = 2 \) (Ising model):

\[ z_{\text{max}}(\nu) = \text{radius of convergence of } G(q, \nu, z). \]

\[ \nu_c = (3 + \sqrt{5})/2 \]

The type of singularity of \( G(2, \nu, z) \) is different at the critical value \( \nu_c \).
Method and results
Warm up: Counting trees

Rooted plane trees with $n$ edges:

$n = 0$ | $n = 1$ | $n = 2$ | $n = 3$
Warm up: Counting trees

Recursive decomposition:
Warm up: Counting trees

Recursive decomposition:

We define the **generating function**: \( T(z) = \sum_{T \text{ tree}} z^{|T|}. \)

Decomposition gives

\[
T(z) = 1 + \sum_{T_1, T_2 \text{ trees}} z \cdot z^{|T_1|} \cdot z^{|T_2|} = 1 + z T(z)^2.
\]
Warm up: Counting trees

Recursive decomposition:

We define the **generating function**: \( T(z) = \sum_{T \text{ tree}} z^{|T|} \).

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T(z) = 1 + zT(z)^2
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\( T(z) \) is **algebraic**: of the form \( \text{Pol}(T(z), z) = 0 \).

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⇒ Number of trees with \( n \) edges \( \sim k n^{-3/2} R^n \).

The exponent \(-3/2\) is determined by the type of singularity of \( T(z) \).
Warm up: Counting trees

Recursive decomposition:

\[ T(z) = 1 + zT(z)^2 \]

We define the generating function: \( T(z) = \sum_{T \text{ tree}} z^{|T|} \).

⇒ \( T(z) \) is algebraic: of the form \( \text{Pol}(T(z), z) = 0 \).

⇒ Number of trees with \( n \) edges \( \sim kn^{-3/2}R^n \).

Even better, using Lagrange inversion formula:
⇒ Number of trees with \( n \) edges \( \text{Cat}_n = \frac{1}{n+1} \binom{2n}{n} \).
### Method - comparative study

<table>
<thead>
<tr>
<th>Trees</th>
<th>Maps (without Potts model)</th>
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<tbody>
<tr>
<td><img src="image" alt="Tree Diagram" /></td>
<td><img src="image" alt="Map Diagram" /></td>
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</tbody>
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**Generating function**

- Trees: \( T = \sum_{T \text{ tree}} z^\#\text{edges} \)
- Maps: \( M(x) = \sum_{M \text{ map}} x^{\deg_0 z^\#\text{edges}} M(x) \)

**Functional equation with catalytic variable \( x \)**

- Trees: \( T = 1 + z T^2 \)
- Maps: \( M(x) = 1 + x^2 z M(x)^2 + xz \frac{x M(x) - M(1)}{x - 1} \)

**“Quadratic Method”**

- Maps: \( M(1) = \sum_{M \text{ map}} z^\#\text{edges} \)

**Algebraic equation**

- Trees: \( 1 - 16z + (18z - 1)M(1) - 27z^2 M(1)^2 = 0 \)
Generating function

\[ G(x, y) \equiv G(x, y; q, \nu, z) = \sum_{M \text{ map}} x^{\deg_0} y^{\deg_1} P(q, \nu) z^{\#\text{edges}} \]

\[ G(x, y) = 1 + (q-1+u)x^2yzG(x, y)G(x, 1) + uxy^2zG(x, y)G(1, y) \]

\[ + \left[ xyz \frac{xG(x, y)-G(1, y)}{x-1} - xyzG(x, y)G(1, y) \right] \]

\[ +(u-1) \left[ xyz \frac{xG(x, y)-G(x, 1)}{y-1} - xyzG(x, y)G(x, 1) \right] \]

functional equation with catalytic variables \(x, y\) \([\text{Tutte 71}]\)
Method

Recursive decomposition of colored maps

Classical generatingfunctionology

Functional equation for $G(x, y; q, \nu, z)$ with catalytic variables $x, y$.

(a.k.a. loop equations)

Method of invariants (core of our approach)

For $q = 2 + 2 \cos(k\pi/m)$, functional equation for $G(1, y; q, \nu, z)$

Generalization of “Quadratic Method” [Jehanne,Bousquet-Mélou 06]

Algebraic equations dependent on $q$

Uniformization via some specializations $y = Y_i(z)$

Differential equation independent of $q$. 
Method

\[ G(x, y) = 1 + (q-1+u)x^2yzG(x, y)G(x, 1) + uxy^2zG(x, y)G(1, y) \]
\[ + \left[ xyz \frac{xG(x,y)-G(1,y)}{x-1} - xyzG(x, y)G(1, y) \right] \]
\[ + (u-1) \left[ xyz \frac{xG(x,y)-G(x,1)}{y-1} - xyzG(x, y)G(x, 1) \right] \]

functional equation with catalytic variables x, y

Method of invariants (core of our approach) (for \( q = 2 + 2 \cos(k\pi/m) \))

\[ D^{m/2}T_m \left( \frac{N}{2\sqrt{D}} \right) = \sum_{r=0}^{m} C_r(z)I^r, \]

where

\[ D = (q\nu+(\nu-1)^2)I^2 - q(\nu+1)I + z(q-4)(\nu-1)(q+\nu-1) + q, \]
\[ N = (4-q)(1/y-1)(\nu-1)+(q+2\nu-2)I - q \]
\[ I = yzqG(1, y) + \frac{y-1}{y} + \frac{zy}{y-1} \]

functional equation with catalytic variable y

Generalization of “Quadratic Method”

Algebraic equations dependent on \( q \)

Differential equation independent of \( q \)
Results about the Potts model on maps

\[ G(q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu) z^\#\text{edges}. \]

Theorem [Bernardi, Bousquet-Mélou]:

- The series \( G(q, \nu, z) \) is **algebraic** whenever \( q = 2 + 2 \cos(k \pi/m) \) and \( q \neq 0, 4 \).

That is, for \( q = \in \{1, 2, 3, 2 + \sqrt{2}, 2 + \sqrt{3}, \ldots\} \), there is an equation of the form \( \text{Poly}_q(G(q, \nu, z), \nu, z) = 0 \).
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\[ G(q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu)z^{|\text{edges}|}. \]

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- Some **explicit algebraic equations** for \( q = 2, 3 \).
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Example: For \( q = 2 \) (Ising model) we get:

\[
G(2, \nu, z) = \frac{1 + 3\nu S - 3\nu S^2 - \nu^2 S^3}{(1 - 2S + 2\nu^2 S^3 - \nu^2 S^4)^2} \times \left( \nu^3 S^6 + 2\nu^2 (1-\nu)S^5 + \nu (1-6\nu S^4 - \nu (1-5\nu) S^3 + (1+2\nu) S^2 - (3+\nu)) \right),
\]

where \( S = z + O(z^2) \) is the series satisfying

\[
S = z \frac{(1 + 3\nu S - 3\nu S^2 - \nu^2 S^3)^2}{1 - 2S + 2\nu^2 S^3 - \nu^2 S^4}.
\]
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- Some **explicit algebraic equations** for \( q = 2, 3 \).

\[ z^n G(2, \nu, z) \sim c_\nu n^{-5/2} R^n_\nu \text{ if } \nu \neq \nu_c, \]
\[ z^n G(2, \nu, z) \sim c_\nu n^{-7/3} R^n_\nu \text{ if } \nu = \nu_c. \]

(matching the physicists’ predictions)
Results about the Potts model on maps

\[ G(q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu) z^{\#\text{edges}}. \]

**Theorem [Bernardi, Bousquet-Mélou]:**

- There is a system of *differential equations* valid for all \( q \).
Results about the Potts model on maps

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**Theorem [Bernardi, Bousquet-Mélou]:**

- There is a system of **differential equations valid for all** \( q \).

There exists \( P(z, t) = \sum_{i=0}^{4} P_i t^i, \quad Q(z, t) = \sum_{i=0}^{2} Q_i t^i, \quad R(z, t) = \sum_{i=0}^{2} R_i t^i \) with \( P_i, Q_i, R_i \in \mathbb{Q}[q, \nu][[z]] \) such that

\[
\frac{1}{Q} \frac{\partial}{\partial z} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial t} \left( \frac{R^2}{PD^2} \right).
\]

with \( D = (q\nu + (\nu - 1)^2)t^2 - q(\nu + 1)t + z(q - 4)(\nu - 1)(q + \nu - 1) + q \)

and \( P_4 = 1, \quad R_2 = 3 - \nu - q, \quad P(0, t) = t^2(t - 1)^2, \quad Q(0, t) = t(t - 1) \).

These series are unique and **determine** \( G(q, \nu, z) \) through:

\[
12 z^2(q\nu + (\nu - 1)^2)G(q, \nu, z) = \]

\[
P_2 - 2 Q_0 - 8 z(3 - \nu - q)Q_1 - Q_1^2 + 4 z(3\nu - 2 + q) - 12 z^2(3 - \nu - q)^2.
\]
Results about the Potts model on Triangulations

\[ G(q, \nu, z) = \sum_{M \text{ triangulation}} P_M(q, \nu) z^{\#\text{edges}}. \]

**Theorem [Bernardi, Bousquet-Mélou]:**

- The series \( G(q, \nu, z) \) is **algebraic** whenever \( q = 2 + 2 \cos(k\pi/m) \) and \( q \neq 0, 4 \).
- Some **explicit algebraic equations** for \( q = 2, 3 \).
- There is a system of **differential equations** valid for all \( q \).
Other results:

- For **planar maps**, [Guionnet, Jones, Shlyakhtenko, Zinn-Justin 2012] give a **functional equation** for $G(q, \nu, z)$ in terms of Theta functions.
Denote $\alpha$ and $\beta$ the weight of monochromatic and dichromatic edges and denote $\delta = q^2 = 2\cos(\sigma \pi)$ with $\sigma \in \mathbb{R}$, and denote $C(\delta, \alpha, \beta) = G(q, \nu, z)$.

Then the series $C(\gamma; \delta, \alpha, \beta)$ with “catalytic variable” $\gamma$ is given in terms of a series $M(\gamma)$ by:

$$C(\gamma; \delta, \alpha, \beta) = \frac{h(\gamma)\alpha}{\gamma} M(h(\gamma)),$$

where $h(\gamma)$ is the inverse of the function $g(z) = \frac{z\alpha}{1 - z^2M(z)}$.

Moreover, $M(\gamma)$ and four auxiliary series $a_1 < a_2 < b_1 < b_2$ in $\delta, \alpha, \beta$ are simultaneously defined by:

$$\frac{1}{z(u)} M \left( \frac{\alpha}{z(u)} \right) = \frac{1}{q + q^{-1}} \left( \varphi(u) + \varphi(-u) + \frac{2q z(u)}{(1 - q^2)\alpha} + \frac{(q^2 + 1)(z(u) - 1)}{(1 - q^2)\beta} \right),$$

where

$$\varphi(u) = c_+ \frac{\Theta(u - u_\infty - 2\sigma K)}{\Theta(u - u_\infty)} + c_- \frac{\Theta(u + u_\infty - 2\sigma K)}{\Theta(u + u_\infty)},$$

and $z(u)$ is the functional inverse of

$$u(z) = \frac{i}{2} \sqrt{(b_1 - a_1)(b_2 - a_2)} \int_{b_2}^z \frac{dx}{\sqrt{(x - a_1)(x - a_2)(x - b_1)(x - b_2)}},$$

and $\Theta(u)$ denotes a Theta function with periods $K, K'$, and $K, K', u_\infty, z_\infty$, $c_\pm$ are constant given in terms of $a_1, b_1, \alpha, \beta, \delta$. 

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- For **properly colored triangulations** [Tutte 73-84] gives a **differential equation** for $H = G(q, 0, z)$:

\[
2q^2(1-q)z + (qz + 10H - 6zH')H'' + q(4-q)(20H - 18zH' + 9z^2H'') = 0.
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This is our guide, but the proof is long!

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases \( \lambda = 1 \) and \( \lambda = 2 \).

[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case \( \lambda = \tau + 1 \).

[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case \( \lambda = 3 \).

[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case \( \lambda = \infty \).

[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations.

[Tutte 78] On a pair of functional equations of combinatorial interest.

[Tutte 82] Chromatic solutions.

[Tutte 82] Chromatic solutions II.


Summarized in: [Tutte 95] Chromatic sums revisited.
Open questions

Question 1. Can we translate the system of differential equations to just one differential equation?
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**Question 2.** Can we obtain asymptotic growth of coefficients for the other values of $q$?

**Question 3.** Generating function is algebraic for $q = 2 + 2 \cos(k\pi/m)$. Is there a bijection with trees for these values of $q$?

**Theorem [Bernardi, Fusy 12]** For maps without matter, any family defined by degree of faces + girth constraints has an algebraic generating function, and is in bijection with a family of trees (defined by local degree constraints).
Thanks.