Differential equations for colored maps

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Counting colored maps: algebraic equations [JCTB 11]
Counting colored maps: differential equations (coming soon)

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Statistical mechanics on maps
Maps

**Definition:** A **planar map** is a gluing of polygons (pairing of the edges) forming the sphere.
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Equivalently, a **planar map** is an embedding of a connected planar graph considered up to deformation.
Random maps

Choose uniformly randomly a planar map made of \( n = 1000 \) squares. Consider this as a random metric space \((V, d)\).
Random surfaces

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**Theorem.** [Le Gall 2007, Miermont/Le Gall 2012] $Q_n$ converges in distribution (in the Gromov Hausdorff topology) toward a continuous random metric space, which almost surely

- is **homeomorphic to the sphere**
- has **Hausdorff dimension 4**.
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The Brownian map is **universal**: it is also the limit of uniformly random triangulations, uniformly random simple maps, etc.
Maps + statistical mechanics model.

Example: spanning-tree model.
configuration = map + spanning tree.
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Motivation 1: Knizhnik-Polyakov-Zamolodchikov-formula:
Conjectural relation between critical exponents in regular lattices Vs random lattices.
Maps + statistical mechanics model.

Motivation 2: Different probability measure on maps. Choose maps of size $n$ with probability proportional to partition function.
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Example: For spanning-tree model, choose maps with probability proportional to number of spanning trees.
Maps + statistical mechanics model.

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Hausdorff dimension of spanning-tree random map is $D_H \approx 3.5(5)$. 
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**Easier information: Counting exponent.**

Number of (rooted) maps with $n$ edges is $M_n \sim k n^{-5/2} R^n$.

Number of maps+spanning trees: $T_n \sim \kappa n^{-3} \rho^n$
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**Theorem [Bender, Canfield 94, Bernardi, Fusy 12]**
**Counting exponent** $\alpha = 5/2$ is universal for maps without model:
same for any map family defined by face degree and girth constraints.
**Counting exponent** $\alpha = 3$ is also universal for maps + spanning tree.
Maps + statistical mechanics model.

Prediction from physics:
\[
\alpha = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12}, \quad D_H = 2 \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}}
\]
where \( c \) is the central charge
\((c = 0 \text{ for maps without model, } c = -2 \text{ for spanning-tree model, } \ldots)\).
Geometric interpretation for counting exponent?

Suppose number of objects of size $n$ is $M_n \sim k n^{-\alpha} R^n$. Sample a pair of objects of total size $n$. How small is the smaller one?
Geometric interpretation for counting exponent?

Suppose number of objects of size $n$ is $M_n \sim kn^{-\alpha}R^n$.

Sample a pair of objects of total size $n$. How small is the smaller one?

Smaller one has size $s$ with probability proportional to

$$\approx ks^{-\alpha}R^s \cdot k(n - s)^{-\alpha}R^{n-s} = k^2n^{2\alpha}R^n \cdot \left(\frac{s}{n} \left(1 - \frac{s}{n}\right)\right)^{-\alpha}.$$  

Thus, $\alpha$ large tends to make $\frac{s}{n}$ small.
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**Conclusion:** the counting exponent tells something about fractal properties of random objects.
Potts model on maps
Potts model (parameters $q$, $\nu$)

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The **partition function of the Potts model** on the graph $G$ is

$$P_G(q, \nu) = \sum_{q\text{-coloring}} \nu \# \text{monochromatic edges}. $$
Potts model

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- The case \( q = 2 \) is the Ising model (without external field).
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- \( P_G(q, \nu) \) satisfies a recurrence relation:

\[ P_G(q, \nu) = P_{G \backslash e}(q, \nu) + (\nu - 1) P_{G / e}(q, \nu). \]
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- \( P_G(q, \nu) \) is a polynomial in the variables \( q, \nu \)
  It is equivalent to the **Tutte polynomial** of \( G \).
  (a.k.a. partition function of the **FK-Cluster model**).
The Potts model on planar maps

The **partition function** of the **Potts model on maps** is

\[
G(q, \nu, z) = \sum_{M, \text{map}} P_M(q, \nu) z^{\#\text{edges}}.
\]

This is a series in \( z \) with coefficients polynomial in \( q, \nu \).
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- Characterize the partition function \( G(q, \nu, z) \).
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Goals:

- Characterize the partition function \( G(q, \nu, z) \).
- Study the phase transitions

  Counting exponent: \( [z^n]G(q, \nu, z) = kn^{-\alpha} R^n \)

  The exponent \( \alpha \) depends on the type of singularity of \( G(q, \nu, z) \) at its radius of convergence.
Results

All maps / Triangulations
Results for Potts on all maps

\[ G(q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu)z^{\#\text{edges}}. \]

**Theorem [Bernardi, Bousquet-Mélou]:**

- The series \( G(q, \nu, z) \) is **algebraic** whenever \( q = 2 + 2 \cos(k\pi/m) \) and \( q \neq 0, 4 \).

That is, for \( q \in \{1, 2, 3, 2 + \sqrt{2}, 2 + \sqrt{3}, \ldots\} \), there is an equation of the form \( \text{Poly}_q(G(q, \nu, z), \nu, z) = 0 \).
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Example: For \( q = 2 \) (Ising model) we get:

\[ G(2, \nu, z) = \frac{1 + 3\nu S - 3\nu S^2 - \nu^2 S^3}{(1 - 2S + 2\nu^2 S^3 - \nu^2 S^4)^2} \]
\[ \times \left( \nu^3 S^6 + 2\nu^2 (1-\nu) S^5 + \nu (1-6\nu S^4 - \nu (1-5\nu) S^3 + (1+2\nu) S^2 - (3+\nu)) \right), \]

where \( S = z + O(z^2) \) is the series satisfying

\[ S = z \frac{(1 + 3\nu S - 3\nu S^2 - \nu^2 S^3)^2}{1 - 2S + 2\nu^2 S^3 - \nu^2 S^4}. \]
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- Some **explicit algebraic equations** for \( q = 2, 3 \).

\[ q = 2 \]

\[ \nu_c = (3 + \sqrt{5})/2 \]

\[ z_{\max}(\nu) = \text{radius of convergence} \]

\[ [z^n]G(2, \nu, z) \sim c_\nu n^{-5/2} R_\nu^n \text{ if } \nu \neq \nu_c, \]

\[ [z^n]G(2, \nu, z) \sim c_\nu n^{-7/3} R_\nu^n \text{ if } \nu = \nu_c. \]

Type of singularity of \( G(q, \nu, z) \) is different at critical value \( \nu_c \).

\[ \Rightarrow \] different counting exponent \( \alpha \Rightarrow \) different type of random maps.
Results for Potts on all maps

\[ G(q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu) z^\# \text{edges}. \]

**Theorem [Bernardi, Bousquet-Mélou]:**
- There is a system of **differential equations** valid for all \( q \).
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**Theorem [Bernardi, Bousquet-Mélou]:**

- There is a system of differential equations valid for all \( q \).

There exists \( P(y) = \sum_{i=0}^{4} P_i y^i \), \( Q(y) = \sum_{i=0}^{2} Q_i y^i \), \( R(y) = \sum_{i=0}^{2} R_i y^i \) with \( P_i, Q_i, R_i \in \mathbb{Q}[q, \nu][[z]] \) such that

\[ \frac{1}{Q} \frac{\partial}{\partial z} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{PD^2} \right). \]

with \( D = (q\nu + (\nu - 1)^2)y^2 - q(\nu + 1)y + z(q - 4)(\nu - 1)(q + \nu - 1) + q \)
and \( P_4 = 1, \ R_2 = 3 - \nu - q, \ [z^0]P(y) = y^2(y - 1)^2, \ [z^0]Q(y) = y(y - 1). \)

The series \( P_i, Q_i, R_i \) are uniquely determined by this system, and

\[ G(q, \nu, z) = \frac{P_2 - 2Q_0 - 8z(3-\nu-q)Q_1 - Q_1^2 + 4z(3\nu - 2 + q) - 12z^2(3-\nu-q)^2}{12z^2(q\nu + (\nu - 1)^2)}. \]
Results for Potts on triangulations

\[ T(q, \nu, z) = \sum_{M \text{ triangulation}} P_M(q, \nu) z^{\# \text{vertices}}. \]

Theorem [Bernardi, Bousquet-Mélou]:

- The series \( T(q, \nu, z) \) is algebraic whenever \( q = 2 + 2 \cos(k\pi/m) \) and \( q \neq 0, 4 \).
- Some explicit algebraic equations for \( q = 2, 3 \).
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- Some explicit algebraic equations for \( q = 2, 3 \).
- There is a system of differential equations valid for all \( q \).
- Integration in special cases \((\nu = 0, q = 0, q = 4)\)

\[ S = T(4, \nu, z) \text{ satisfies } P(S, S_z, S_{zz}) = 0 \]

where

\[
\begin{align*}
P(X, Y, Z) &= 3584 \nu^4 \beta^2 ZY^4 w^2 - 1024 \nu^4 (160X\nu^2 - \delta^2) ZY^2 w^2 + 3\beta^5 \delta Z^2 Y^4 w \\
&- 5\beta^5 \delta ZX^2 Y^3 + 5(160X\nu^2 - \delta^2) X \beta^2 (20X\nu^2 + \delta^2) Z^2 \\
&- 8\beta^4 (10X\nu^2 - \delta^2) Z^2 Y^3 w - 15\beta^4 \delta^2 XZ^2 Y^2 - 6144 \nu^4 (320X\nu^2 + \delta^2) w^3 Z^2 Y \\
&+ 48\beta^4 \nu^2 Z^2 Y^4 w^2 + 10240 \nu^4 \beta^2 Z^2 Y^3 w^3 - 40\beta^4 \nu^2 ZY^4 X + 4096 Y^4 \nu^6 w^2 \\
&- 8\nu^2 (160X\nu^2 - \delta^2) \beta \delta ZY^2 w - 138240 X \nu^4 \beta \delta w^2 Z^2 Y - 2160 \nu^2 X \beta^2 \delta^2 w Z^2 Y \\
&+ 1536 \nu^4 \beta \delta ZY^3 w^2 - 24 \nu^2 \beta^2 (320X\nu^2 - \delta^2) ZY^3 w + 20 \nu^2 (160X\nu^2 - \delta^2) X \beta^2 ZY^2 \\
&+ 40 \beta^3 \nu^2 \delta ZY^4 w + 768 \nu^2 \beta^3 Z^2 Y^3 w^2 + 49152 \nu^4 \beta \delta Z^2 Y^2 w^3 \\
&- 288 \beta^2 \nu^2 (80X\nu^2 - 3\delta^2) Z^2 Y^2 w^2 - 6 \beta^3 (280X\nu^2 - \delta^2) Z^2 Y^2 w \\
&+ 15 X \delta \beta^3 (40X\nu^2 - \delta^2) Z^2 Y + (160X\nu^2 - \delta^2) \beta \delta (560X\nu^2 + \delta^2) Z^2 w - 6, \\
\end{align*}
\]

with \( \beta = \nu - 1 \) and \( \delta = \nu - 2 \).
Intermediate result: functional equation

\[ G(y) = G(y; q, \nu, z) = \sum_{M \text{ map}} P_M(q, \nu) z^\#\text{edges} y^\text{outer degree}. \]

Theorem [Bernardi, Bousquet-Mélou]:

For \( q = 2 + 2 \cos(k\pi/m) \), with \( k, m \) integers,

\[
D(I(y))^{m/2} T_m \left( \frac{N(I(y), y)}{2\sqrt{D(I(y))}} \right) = \sum_{r=0}^m C_r I(y)^r,
\]

where

\[
I(y) = yzqG(y) + \frac{y-1}{y} + \frac{zy}{y-1}
\]

\[
T_m = m^{\text{th}} \text{Chebyshev polynomial},
\]

\[
N(\alpha, y) = (4-q)(1/y-1)(\nu-1) + (q+2\nu-2)\alpha - q,
\]

\[
D(\alpha) = (q\nu + (\nu-1)^2)\alpha^2 - q(\nu+1)\alpha + z(q-4)(\nu-1)(q+\nu-1) + q.
\]

and \( C_r \) are unknown series in \( \mathbb{C}[q, \nu][[z]] \), related to \( \frac{\partial^k}{\partial y^k} G(1) \).
Alternative results:

- For **all maps**, [Guionnet, Jones, Shlyakhtenko, Zinn-Justin 2012] give a **functional equation** for $G(q, \nu, z)$. 
Denote $\alpha$ and $\beta$ the weight of monochromatic and dichromatic edges, denote $\delta = \sqrt{q} = p + p^{-1}$ with $p = e^{i\pi\sigma}$, and denote $C(\delta, \alpha, \beta) = G(q, \nu, z)$.

Then the series $C(\gamma; \delta, \alpha, \beta)$ with “extra variable” $\gamma$ is given in terms of a series $h(\gamma)$ by:

$$C(\gamma; \delta, \alpha, \beta) = \frac{\alpha}{\gamma^2 h(\gamma)} - \frac{\alpha^2}{\gamma^2},$$

where $h(\gamma)$ is the functional inverse of the function $\gamma(h) = \frac{\alpha h}{1 - h^2 M(h)}$

where, $M(x)$ and four auxiliary series $a_1 < a_2 < b_1 < b_2$ in $\delta, \alpha, \beta$ are simultaneously defined by:

$$\frac{1}{z(u)} M \left( \frac{\alpha}{z(u)} \right) = \frac{1}{p - p^{-1}} \left( \varphi(u) + \varphi(-u) + \frac{2p z(u)}{(1 - p^2)\alpha} + \frac{(p^2 + 1)(z(u) - 1)}{(1 - p^2)\beta} \right),$$

$$M(1/a_1) = 0, \quad M(1/a_2) = 0, \quad \tilde{M}(1/b_1) = 0, \quad \tilde{M}(1/b_2) = 0,$$

where

$$\varphi(u) = c_+ \frac{\Theta(u - u_\infty - 2\sigma K)}{\Theta(u - u_\infty)} + c_- \frac{\Theta(u + u_\infty - 2\sigma K)}{\Theta(u + u_\infty)},$$

and $z(u)$ is the functional inverse of

$$u(z) = \frac{i}{2} \sqrt{(b_1 - a_1)(b_2 - a_2)} \int_{b_2}^{z} \frac{dx}{\sqrt{(x - a_1)(x - a_2)(x - b_1)(x - b_2)}},$$

and $\Theta(u)$ denotes a Theta function with periods $K, K'$, and $K, K', u_\infty, z_\infty, c_+, c_-$ are constants given in terms of $a_1, b_1, \alpha, \beta, \delta$. 
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- [Borot, Bouttier, Guitter 2012] Obtained the same results using a different approach (gasket decomposition of related loop-model).
Alternative results:

- For all maps, [Guionnet, Jones, Shlyakhtenko, Zinn-Justin 2012] give a functional equation for $G(q, \nu, z)$.

- [Borot, Bouttier, Guitter 2012] Obtained the same results using a different approach (gasket decomposition of related loop-model).

- [Bonnet, Eynard 99] gave similar results for triangulations (incomplete proof).
Method
Our guide:

William Tutte
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For properly colored triangulations [Tutte 73-84] obtained a differential equation for $H = G(q, 0, z)$:

$$2q^2(1-q)z + (qz + 10H - 6zH')H'' + q(4-q)(20H - 18zH' + 9z^2H'') = 0.$$
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\[
2q^2(1-q)z + (qz + 10H - 6zH')H'' + q(4-q)(20H - 18zH' + 9z^2H'') = 0.
\]

But the proof is long...

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases \( \lambda = 1 \) and \( \lambda = 2 \).
[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case \( \lambda = \tau + 1 \).
[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case \( \lambda = 3 \).
[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case \( \lambda = \infty \).
[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations.
[Tutte 78] On a pair of functional equations of combinatorial interest.
[Tutte 82] Chromatic solutions.
[Tutte 82] Chromatic solutions II.
Summarized in: [Tutte 95] Chromatic sums revisited.
Warm up: Counting trees

Rooted plane trees with $n$ edges:

\[ n = 0 \quad | \quad n = 1 \quad | \quad n = 2 \quad | \quad n = 3 \]
Warm up: Counting trees

Recursive decomposition:
Warm up: Counting trees

Recursive decomposition:

$T = \sum_{\tau \text{ tree}} z^{\#\text{edges}}.$

We define the **generating function**: $T = \sum_{\tau \text{ tree}} z^{\#\text{edges}}.$
Warm up: Counting trees

Recursive decomposition:

We define the generating function: \( T = \sum_{\tau \text{ tree}} z^{\# \text{edges}} \).

\[
T = 1 + zT^2
\]

\( \Rightarrow \) \( T \) is algebraic: of the form \( \text{Pol}(T, z) = 0 \).
Method - comparative study

Trees

Generating function

\[ T = \sum_{T \text{ tree}} z^{\# \text{edges}} \]

\[ T = 1 + z T^2 \]

algebraic equation

Maps

Generating function

\[ M(y) = \sum_{M \text{ map}} y^{\text{out-deg}} z^{\# \text{edges}} \]

\[ M(y) = 1 + y^2 z M(y)^2 + yz \left( \frac{y M(y) - M(1)}{y - 1} \right) \]

functional equation with catalytic variable \( y \).

“Quadratic Method”

\[ \rightarrow M_1 = M(1) = \sum_{M \text{ map}} z^{\# \text{edges}} \]

\[ 1 - 16z + (18z - 1) M_1 - 27z^2 M_1^2 = 0 \]

algebraic equation
Method - comparative study

Maps with Potts model

Generating function

\[ G(x, y) \equiv G(x, y; q, \nu, z) = \sum_{M \text{ map}} x^{\deg_0} y^{\deg_1} P(q, \nu) z^{\#\text{edges}} \]

\[ G(x, y) = 1 + (q - 1 + \nu)x^2 yz G(x, y) G(x, 1) + \left[ xyz \frac{x G(x, y) - G(1, y)}{x - 1} - xyz G(x, y) G(1, y) \right] \]

\[ + \nu xy^2 z G(x, y) G(1, y) + (\nu - 1) \left[ xyz \frac{x G(x, y) - G(x, 1)}{y - 1} - xyz G(x, y) G(x, 1) \right] \]

functional equation with two catalytic variables \( x, y \)
Method - comparative study

Maps with Potts model

Generating function $G(x, y) = G(x, y; q, \nu, z)$

Functional equation for $G(x, y)$. Catalytic variables $x, y$.

For $q = 2 + 2 \cos(k\pi/m)$ Method of invariants (core of our approach)

Functional equation for $G(1, y)$: Catalytic variable $y$.

Generalization of “Quadratic Method”

Algebraic equations dependent on $q$

Uniformization via some specializations $y = Y_i(z)$

Differential equation independent of $q$. 
Overview

\[ G(x, y) = 1 + (q-1+\nu)x^2yzG(x, y)G(x, 1) + \left[ xyz \frac{xG(x, y)-G(1, y)}{x-1} - xyzG(x, y)G(1, y) \right] \]

\[ + \nu xy^2zG(x, y)G(1, y) + (\nu-1) \left[ xyz \frac{xG(x, y)-G(x, 1)}{y-1} - xyzG(x, y)G(x, 1) \right] \]

functional equation

Method of invariants For \( q = 2 + 2 \cos(k\pi/m) \)
(core of our approach)

\[ D(I)^{m/2} T_m \left( \frac{N(I, y)}{2\sqrt{D(I)}} \right) = \sum_{r=0}^{m} C_r I^r, \]

where

\[ I = I(y) = yzqG(1, y) + \frac{y-1}{y} + \frac{zy}{y-1} \]

\[ T_m = m^{th} \text{Chebyshev polynomial} \]

\[ N(\alpha, y) = (4-q)(1/y-1)(\nu-1)+(q+2\nu-2)\alpha - q, \]

\[ D(\alpha) = (q\nu+(\nu-1)^2)\alpha^2 - q(\nu+1)\alpha + z(q-4)(\nu-1)(q+\nu-1) + q. \]

algebraic equation depends on \( q \)

\[ G(1, 1) = \frac{P_2 - 2 Q_0 - 8 z(3-\nu-q)Q_1 - Q_2^2 + 4 z(3\nu-2+q) - 12 z^2(3-\nu-q)^2}{12 z^2(q\nu+(\nu-1)^2)} \]

where the polynomials \( P = \sum_{i=0}^{4} P_i \alpha^i, \quad Q = \sum_{i=0}^{2} Q_i \alpha^i, \quad R = \sum_{i=0}^{2} R_i \alpha^i \) are determined by

\[ \frac{1}{Q} \frac{\partial}{\partial z} \left( \frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial \alpha} \left( \frac{R^2}{PD^2} \right) \quad \text{and} \quad P_4 = 1, \quad R_2 = 3-\nu-q, \quad [t^0]P = \alpha^2(\alpha-1)^2, \quad [t^0]Q = \alpha(\alpha-1). \]
Simpler case: bipartite maps. \( M(y) = \sum_{M \text{ bipartite map}} y^{\text{deg}_0/2} z^{\#\text{edges}} \)

Functional equation with catalytic variable \( y \)

\[
M(y) = 1 + yzM(y)^2 + zy \frac{M(y) - M(1)}{y-1}
\]
Simpler case: bipartite maps. \( M(y) = \sum_{M \text{ bipartite map}} y^{\text{deg}_0/2} z^{\#\text{edges}} \)

Functional equation with catalytic variable \( y \):

\[
M(y) = 1 + yzM(y)^2 + zy \frac{M(y) - M(1)}{y-1}
\]

Form square:

\[
A(y) = (2zy(y-1)M(y) + 1-y + zy)^2 = (1 - y + zy)^2 - 4zy(1-y)(1-y + zyM_1)
\]
Simpler case: bipartite maps. \( M(y) = \sum_{M \text{ bipartite map}} y^{\deg_0/2} \cdot z \cdot \#\text{edges} \)

\[
M(y) = 1 + yzM(y)^2 + zy \frac{M(y)-M(1)}{y-1}
\]

functional equation with catalytic variable \( y \)

Form square:

\[
(2zy(y-1)M(y)+1-y+zy)^2 = (1 - y + zy)^2 - 4zy(1-y)(1-y+z My_1)
\]

there exists a series \( \exists Y = 1 + t + 3t^2 + \ldots \) such that \( A(Y) = 0 \).
Simpler case: bipartite maps. $M(y) = \sum_{M \text{ bipartite map}} y^{\deg_0/2} z^{\#\text{edges}}$

$$M(y) = 1 + yz M(y)^2 + zy \frac{M(y) - M(1)}{y - 1}$$

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$$A(y) = (2zy(y-1)M(y) + 1 - y + zy)^2 = (1 - y + zy)^2 - 4zy(1-y)(1-y+zyM_1)$$

$$B(y)$$

There exists a series $\exists Y = 1 + t + 3t^2 + \ldots$ such that such $A(Y) = 0$. This series is root of $B(y)$ and $B_y(y)$. 
Functional equations $\rightarrow$ Algebraic equations

Simpler case: bipartite maps. $M(y) = \sum_{M \text{ bipartite map}} y^{\text{deg}_0/2} z^{\#\text{edges}}$

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Functional equation with catalytic variable $y$

Form square:

$$A(y) = (2zy(y-1)M(y) + 1-y+zy)^2 = (1-y+zy)^2 - 4zy(1-y)(1-y+zyM_1)$$

$$B(y)$$

There exists a series $\exists Y = 1 + t + 3t^2 + \ldots$ such that such $A(Y) = 0$. This series is root of $B(y)$ and $B_y(y)$.

Hence the discriminant of $B(y)$ is 0:

$$16z^2 M_1^2 + (1 - 12z - 8z^2)M_1 + z^2 + 11z - 1 = 0.$$
Functional equations $\rightarrow$ Algebraic equations

We apply a generalization [MBM-Jehanne 06] of the quadratic method to:

For $q = 2 + 2 \cos(k\pi/m)$

$$ D(I(y))^{m/2} T_m \left( \frac{N(I(y), y)}{2\sqrt{D(I(y))}} \right) = \sum_{r=0}^{m} C_r I(y)^r, $$

where

$$ I(y) = yzqG(1, y) + \frac{y-1}{y} + \frac{zy}{y-1} $$

$$ T_m = m^\text{th} \text{Chebyshev polynomial}, $$

$$ N(\alpha, y) = (4-q)(1/y-1)(\nu-1)+(q+2\nu-2)\alpha - q, $$

$$ D(\alpha) = (q\nu+(\nu-1)^2)\alpha^2-q(\nu+1)\alpha+z(q-4)(\nu-1)(q+\nu-1) + q; $$

Unknown series of $z$

functional equation with catalytic variable $y$

**Differences:** Not quadratic. Has $m$ unknown series $C_r$ instead of 1.

$\rightarrow$ System of $3m$ algebraic equations. Always solvable in theory, but not in practice except for small $m$. 
Functional equations $\longrightarrow$ Differential equations
Simpler case: bipartite maps. \( M(y) = \sum_{M \text{ bipartite map}} y^{\text{deg}_0/2} z \#\text{edges} \)

\[
M(y) = 1 + yz M(y)^2 + zy \frac{M(y) - M(1)}{y-1}
\]

Functional equation with catalytic variable \( y \)
Simpler case: bipartite maps. \( M(y) = \sum_{M\text{ bipartite map}} y^{\text{deg}_0/2} z^\#\text{edges} \)

\[ M(y) = 1 + yz M(y)^2 + zy \frac{M(y) - M(1)}{y - 1} \]

Form square:

\[ A(y) = (2yz(y - 1)M(y) + 1 - y + zy)^2 \]

\[ B(y) = (1 - y + zy)^2 - 4zy(1 - y)(1 - y + zyM_1) \]
Simpler case: bipartite maps. \[ M(y) = \sum_{M \text{ bipartite map}} y^{\deg_0/2} z^{\#\text{edges}} \]

\[ M(y) = 1 + yz M(y)^2 + zy \frac{M(y) - M(1)}{y - 1} \]

Functional equation with catalytic variable \( y \)

Form square:

\[ (2zy(y-1)M(y)+1-y+zy)^2 = (1-y+zy)^2 - 4zy(1-y)(1-y+zyM_1) \]

\( A(y) \) \hspace{1cm} \( B(y) \)

The series \( Y \) cancelling \( A(y) \) is root of \( B(y) \), \( B_y(y) \) and \( B_z(y) \), hence there exists polynomials \( P(y), Q(y), R(y) \) of degree 1, 1, 2 such that

\[ B(y) = (y - Y)^2 P(y), \quad B_y(y) = (y - Y) Q(y), \quad B_z(y) = (y - Y) R(y). \]
Functional equations $\rightarrow$ Differential equations

Simpler case: bipartite maps. $M(y) = \sum_{M \text{ bipartite map}} y^{\deg_0/2} z^{\#\text{edges}}$

$$M(y) = 1 + yz M(y)^2 + zy \frac{M(y) - M(1)}{y-1}$$

functional equation with catalytic variable $y$

Form square:

$$\begin{align*}
A(y) &= (2zy(y-1)M(y)+1-y+zy)^2 = (1-y+zy)^2 - 4zy(1-y)(1-y+zyM_1) \\
B(y) &= \frac{1}{Q} \frac{\partial}{\partial z} \left( \frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left( \frac{R^2}{P} \right)
\end{align*}$$

The series $Y$ cancelling $A(y)$ is root of $B(y)$, $B_y(y)$ and $B_z(y)$, hence there exists polynomials $P(y), Q(y), R(y)$ of degree 1, 1, 2 such that

$$B(y) = (y-Y)^2 P(y), \quad B_y(y) = (y-Y) Q(y), \quad B_z(y) = (y-Y) R(y).$$

$$\Rightarrow$$
Functional equations \(\rightarrow\) Differential equations

We apply a generalization of this method to:

For \(q = 2 + 2 \cos(k\pi/m)\)

\[
D(I(y))^{m/2} T_m\left(\frac{N(I(y), y)}{2\sqrt{D(I(y))}}\right) = \sum_{r=0}^{m} C_r I(y)^r,
\]

where

- \(I(y) = yzqG(1, y) + \frac{y-1}{y} + \frac{zy}{y-1}\)
- \(T_m = m^{\text{th}}\) Chebyshev polynomial,
- \(N(\alpha, y) = (4-q)(1/y-1)(\nu-1)+(q+2\nu-2)\alpha - q,\)
- \(D(\alpha) = (q\nu+(\nu-1)^2)\alpha^2 - q(\nu+1)\alpha + z(q-4)(\nu-1)(q+\nu-1) + q,\)

unknown series of \(z\)

Differences:
Not quadratic \(\rightarrow\) many series \(Y_i\).
The RHS is not of the form \(B(y)\) but rather \(B(I(y))\).
Open questions

**Question 1.** Generating function is algebraic for $q = 2 + 2 \cos\left(\frac{k\pi}{m}\right)$. Is there a bijection with trees for these values of $q$?
Open questions

Question 1. Generating function is algebraic for $q = 2 + 2 \cos(k\pi/m)$. Is there a bijection with trees for these values of $q$?

Question 2. What is the combinatorial meaning of the differential equations?

$$2q^2(1-q)z + (qz + 10H - 6zH')H'' + q(4-q)(20H - 18zH' + 9z^2H'') = 0.$$
Thanks.