A short proof of Rayleigh’s Theorem

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Spitzer’s walks in the plane

Walks made of $n$ unit steps.
Direction of each step is uniformly random.
Spitzer’s walks in the plane

Walks made of \( n \) unit steps.
Direction of each step is uniformly random.

Theorem [Rayleigh] The probability for the walk to end at distance less than 1 from the origin is \( \frac{1}{n+1} \).
Spitzer’s walks in the plane

Walks made of $n$ steps of random lengths distributed as $X$. Direction of each step is uniformly random.

**Theorem [Rayleigh]** The probability for the walk to end at distance less than 1 from the origin is $\frac{1}{n+1}$.

**Theorem [B.]** For any positive random variable $X$,

$$\mathbb{P}(X^i > X^j) = \frac{i}{i+j}.$$
Lemma. For any positive random variables $A, B, C$,

$$\mathbb{P}(A > B \ast C) + \mathbb{P}(B > A \ast C) + \mathbb{P}(C > A \ast B) = 1$$
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\]

Proof. We condition on \( A = a, B = b, C = c \) and prove:
\[
\mathbb{P}(a > b \ast c) + \mathbb{P}(b > a \ast c) + \mathbb{P}(c > a \ast b) = 1.
\]

- If \( a \geq b + c \) or \( b \geq a + c \) or \( c \geq a + b \), obvious.

- Otherwise, consider the angles \( \alpha, \beta, \gamma \) of the triangle.
  One has \( \mathbb{P}(a > b \ast c) = \frac{2\alpha}{2\pi} \) etc.

Hence
\[
\mathbb{P}(a > b \ast c) + \mathbb{P}(b > a \ast c) + \mathbb{P}(c > a \ast b) = \frac{\alpha + \beta + \gamma}{\pi} = 1.
\]
Proof.

Lemma. For any positive random variables $A, B, C$,

$$\mathbb{P}(A > B \ast C) + \mathbb{P}(B > A \ast C) + \mathbb{P}(C > A \ast B) = 1$$

Equivalently, $\mathbb{P}(A > B \ast C) + \mathbb{P}(B > A \ast C) = \mathbb{P}(A \ast B > C)$.

Let $P(i, n) = \mathbb{P}(X^i > X^{n-i})$. We want to prove $P(i, n) = \frac{i}{n}$. 
Lemma. For any positive random variables $A, B, C$,
\[ \mathbb{P}(A > B \ast C) + \mathbb{P}(B > A \ast C) + \mathbb{P}(C > A \ast B) = 1 \]
Equivalently, \[ \mathbb{P}(A > B \ast C) + \mathbb{P}(B > A \ast C) = \mathbb{P}(A \ast B > C) \].

Let $P(i, n) = \mathbb{P}(X^i > X^{(n-i)})$. We want to prove $P(i, n) = \frac{i}{n}$.

Apply the lemma to $A = X^i, B = X^j, C = X^{(n-i-j)}$. This gives $P(i, n) + P(j, n) = P(i + j, n)$ whenever $i + j \leq n$.

Thus $nP(1, n) = P(n, n) = 1$.
This gives $P(1, n) = \frac{1}{n}$, and $P(i, n) = i P(1, n) = \frac{i}{n}$. □
The end.