A bijective approach to percolation on random lattices, and maybe more

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Joint work with Nina Holden & Xin Sun - MIT
Model and motivations
Percolation

**Site percolation:** vertices colored *black* with proba $p$
*white* with proba $1 - p$

Site percolation on triangular lattice
Percolation

**Site percolation**: vertices colored *black* with proba $p$
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Percolation

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Questions:
- **Critical probability?**
- **Crossing probabilities?**
Site percolation: vertices colored black with proba $p$ white with proba $1 - p$

Questions:
- Critical probability?
- Crossing probabilities?
- Law of interfaces?
Percolation

**Site percolation**: vertices colored *black* with proba $p$ *white* with proba $1 - p$

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- Crossing probabilities?
- Law of interfaces?
- Mixing properties?
Planar maps

Def. A **planar map** is a way of forming the sphere by gluing polygons.
Planar maps

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Embedded connected planar graph considered up to homeomorphism.
Planar maps

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Def. A **triangulation** is a planar map made of triangles.
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Def. A **triangulation** is a planar map made of triangles.

Def. A map is **rooted** if an edge is marked and oriented.
Percolation on random lattices

We can consider percolation on **random planar triangulations** (instead of regular triangular lattice)
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Same questions:
- Critical probability?
- Crossing probabilities?
- Law of interfaces?
- Mixing properties?

[Curien, Kortchemski 17, Bernardi, Curien, Miermont 18].
Percolation on random lattices

We can consider percolation on random planar triangulations (instead of regular triangular lattice)

Same questions:
- Critical probability?
- Crossing probabilities?
- Law of interfaces?
- Mixing properties?

[Curien,Kortchemski 17, Bernardi, Curien, Miermont 18].

We can also consider infinite random planar triangulations
[Angel, Schramm 04, Angel 03, Angel, Curien 12].
Percolation on random lattices

Is it interesting to consider random lattices?
Percolation on random lattices

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**YES!**

The behavior on regular/random lattices are related (conjecturally)
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Model on random lattices

Model on regular lattice measured against Gaussian free field

Conjecture
Percolation on random lattices

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Model on regular lattice measured against Gaussian free field

Model on random lattices

Conjecture

Would imply KPZ formula [Knizhnik, Polyakov, Zamolodchikov]: Conjecture relating critical exponents regular Vs random lattices.
The bijection
Kreweras walks

**Def.** $\mathcal{K} =$ set of lattice walks on $\mathbb{N}^2$ starting and ending at $(0, 0)$ and made of steps $(1, 0), (0, 1), (-1, -1)$. 
Percolated triangulations

Def. $\mathcal{T} = \text{set of rooted, loopless, planar triangulations, with site-percolation, such that the root-edge goes from white to black.}$
Bijection

Percolated triangulation

3n edges
(n+2 vertices)

Kreweras walks

3n steps
Bijection

Percolated triangulation

$3n$ edges
$(n+2$ vertices)

$2^n \cdot \frac{2^n}{(n+1)(2n+1)} \binom{3n}{n}$

Kreweras walks

$3n$ steps

$\frac{4^n}{(n+1)(2n+1)} \binom{3n}{n}$
Duality

Def. The dual $M^*$ of a map $M$ is obtained by:
Duality

Def. The **dual** \( M^* \) of a map \( M \) is obtained by:
- placing a vertex of \( M^* \) in each face of \( M \)
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Duality: Triangulation $\leftrightarrow$ Cubic map (vertices of degree 3)
          loopless $\leftrightarrow$ bridgeless
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**Duality:**

- Triangulation $\leftrightarrow$ Cubic map (vertices of degree 3)
- loopless $\leftrightarrow$ bridgeless
- Site percolation $\leftrightarrow$ face percolation
Face-percolated cubic maps

Kreweras walks
Exploration tree

**Def.** A spanning tree of a rooted map is a **DFS-tree** if every external edge links a vertex to one of its ancestors.
Exploration tree

**Prop.** For any bridgeless cubic map, there is a **bijection** between
- DFS-tree not containing root
- face-percolation with root having white/black on its left/right.
Exploration tree

**Prop.** For any bridgeless cubic map, there is a **bijection** between
- DFS-tree not containing root
- face-percolation with root having white/black on its left/right.

**From perco to DFS-tree:**
Explore map depth-first by turning left/right at black/white faces.
Exploration tree

**Prop.** For any bridgeless cubic map, there is a *bijection* between
- DFS-tree not containing root
- face-percolation with root having white/black on its left/right.

*From DFS-tree to perco:*
Color white/black the faces at the left/right of tree.
The bijection $\Phi : \mathcal{K} \rightarrow \mathcal{T}$

$aababbacbccabcabcc$
The bijection $\Phi : \mathcal{K} \to \mathcal{T}$

$$aababbacbcacbabc$$
The bijection $\Phi : \mathcal{K} \rightarrow \mathcal{T}$
The bijection $\Phi : K \rightarrow T$

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$\Phi(aababbacbccabcc)$
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$\Phi$

$\mathcal{K}$

$\mathcal{T}$
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$\texttt{aababbacbccabcc}$
The bijection $\Phi : \mathcal{K} \to \mathcal{T}$

$\Phi$:

```
aababbacbcccabcc
```

Diagram: Three axes labeled $a$, $b$, and $c$. A blue path is shown and mapped to a red path labeled $\Phi$. The green arrow points up at the end of the red path.
The bijection $\Phi : \mathcal{K} \rightarrow \mathcal{T}$

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The bijection $\Phi : \mathcal{K} \rightarrow \mathcal{T}$

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$\Phi$

$\mathcal{K}$

$\mathcal{T}$
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The bijection $\Phi : K \to T$

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The bijection $\Phi : \mathcal{K} \to \mathcal{T}$

**Thm.** The map $\Phi$ is a bijection between $\mathcal{K}$ and $\mathcal{T}$.
Variants of the bijection

Spherical case

Disk case

IUPT case
Dictionary: maps $\leftrightarrow$ walks

<table>
<thead>
<tr>
<th>triangulation</th>
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<tr>
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<td>x-coordinate of walk</td>
</tr>
<tr>
<td>vertices</td>
<td>c-steps</td>
</tr>
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<td>c steps of type $abc$</td>
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<td>c-steps of type $abc$ such that $abc$ enclosed by $bc$ parenthesis.</td>
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Dictionary: perco-interface to $\nu \longleftrightarrow$ walk of excursions
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- Empty the bubles
- Shuffle of 2 looptrees
- Flatten each sub-excursion into a single step
- Flattened walk
Dictionary: perco-interface to $v \leftrightarrow$ walk of excursions

Flatten each sub-excursion into a single step
Random 2D geometries
(and some big conjectures)
Random 2D geometries

**SLE curves**

Schramm
Lawler
Werner
Smirnoff
...

**GFF**

Sheffield
Kenyon
Miller
...

**Brownian map**

Schaeffer
Le Gall
Miermont
...
Random 2D geometries

SLE curves

Sheffield
Miller
Werner

... Schramm
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What is... a **SLE curve** (**Schramm–Loewner evolution**)?

A curve $\text{SLE}_\kappa$ is a random (non-crossing) curve in the plane with
- conformal invariance property
- Markov domain property

The parameter $\kappa$ determines how much the curves wiggles.
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SLE curves are proved/conjectured to be scaling limit of curves from statistical mechanics models on regular lattice.
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What is... the **GFF** (Gaussian Free Field)?

**Discrete version:** Let $h(x, y)$ be a function on $\{1, 2, \ldots, n\}^2$ chosen with probability density proportional to

$$\exp \left( - \frac{1}{2} \sum_{(u,v)} (h(u) - h(v))^2 \right).$$

Value of GFF
(image by J. Miller)
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Use \( h \) to define a (random) "density of area" on the square:

\[
\mu_{GFF}(v) := \exp(\gamma h(v)),
\]

for some \( \gamma \in [0, 2] \) (controlling the wildness of \( \mu_{GFF} \)).

GFF area density for \( \gamma = 1.5 \)
(image by J. Miller)
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$$\mu_{GFF}(v) := \exp(\gamma h(v)),$$

for some $\gamma \in [0, 2]$ (controlling the wildness of $\mu_{GFF}$).

Continuous version:

Complications because $h$ no longer a function (only distribution). Can still make sense of $h$ and $\mu_{GFF}(dz) = \exp(\gamma h)dz$.

GFF area density for $\gamma = 1.5$
(image by J. Miller)
What is... the Brownian map?

Take a uniformly random planar triangulation $T_n$ with $n$ triangles of side length $n^{-1/4}$. 
**What is... the Brownian map?**

Take a uniformly random planar triangulation $T_n$ with $n$ triangles of side length $n^{-1/4}$.

**Thm [LeGall, Miermont]:** The random metric space $T_n$ converges in law (for the Gromov Hausdorff topology).

The limit is a random compact metric space (homeomorphic to 2D sphere, but with Hausdorff dimension 4). We call it **Brownian map**.
Peanosphere bijection

Thm [Duplantier, Miller, Sheffield]:
There is a bijection ($\sigma$-algebra preserving map):

\[ \text{GFF} \sqrt{\frac{8}{3}} + \text{CLE}_6 \]
(independent, on the disk)

Brownian motion in 1/3-plane
(from (1,0) to (0,0))
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Brownian motion in 1/3-plane
(from \((1,0)\) to \((0,0)\))

Our bijection is the **exact discrete analogue**!
(but with maps on left-hand side)
Percolation on random triangulations and \( \text{GFF} \sqrt{8/3} + \text{CLE}_6 \)

**Thm [B., Holden, Sun]:** Let \( k > 0 \).

Let \( C_1^n, \ldots, C_k^n \) be the \( k \) longest percolation interfaces in a uniformly random percolated triangulation of size \( n \) (disk topology).

Let \( D_1, \ldots, D_k \) be the \( k \) longest loops in \((\text{GFF} \sqrt{8/3}, \text{CLE}_6)\).

Then, \((\text{length}(C_i^n), \text{area}(C_i^n), \text{pivotal}(C_i^n), \text{pivotal}(C_i^n, C_j^n))_{i,j \in [k]}\)
converges to \((\text{length}(D_i), \text{area}(D_i), \text{pivotal}(D_i), \text{pivotal}(D_i, D_j))_{i,j \in [k]}\).
Percolation on random triangulations and $\text{GFF } \sqrt{\frac{8}{3}} + \text{CLE}_6$

**Proof (sketch):**

- Convergence of walk
- + convergence of loop quantities
Percolation on random triangulations and GFF $\sqrt{\frac{8}{3}} + \text{CLE}_6$

Proof (sketch):

Convergence of walk + convergence of loop quantities
The big conjecture:

random map \rightarrow \text{“natural”} \text{ embedding } \epsilon \rightarrow \text{random density of points (in law)} \rightarrow \text{GFF } \sqrt{\frac{8}{3}}
The big conjecture:

random map

"natural" embedding $\epsilon$

(random density of points)

+ percolation interfaces

$\sqrt{8/3}$

GFF + CLE$_6$
Work in progress [Garban, Gwynne, Holden, Miller, Sepulveda, Sheffield, Sun]

Want to show big conjecture holds for $\epsilon = \textbf{Cardy embedding}$

where $p_i = \mathbb{P}_{\text{perco}}$
Work in progress [Garban, Gwynne, Holden, Miller, Sepulveda, Sheffield, Sun]

Key ingredient needed:

- Same triangulation
- 2 independent percolations

- Same GFF
- 2 independent CLE

2 Kreweras walks

2 Brownian motions
Thanks.