Econ301a: Advanced Microeconomics
Fall 2006
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Midterm Exam: Solutions

Directions: Do all problems. You have 1.5 hours. 100 points total. Stay calm and good luck.

I. True, False, Uncertain (Explain) (7 points each)

1. If a consumer's indirect utility function is given by

$$\frac{g(p_1, p_2, \ldots, p_n)}{y} + h(p_1, p_2, \ldots, p_n)$$

where p_i is the price of good i, and y is income, then the function $g(p_1, p_2, \ldots, p_n)$ must be homogeneous of degree one, and the function $h(p_1, p_2, \ldots, p_n)$ must be homogeneous of degree zero.

True: This problem is pretty clear. The indirect utility is homogeneous of deg zero in (p, y), so the above functions must give us this too. Think about the usual experiment of doubling p and y. Clearly $h()$ must be homogeneous of deg 0, and $g()$ must increase by a factor of 2 to cancel out with the y in the denominator.

2. Consider a world where all prices are constant, except the price of energy. Consumers are one period utility maximizers over a bundle of goods, including energy. All goods are consumed in strictly positive quantities, and prices are also strictly positive. The friendly government announces that it will maintain all consumers at their current utility levels by covering all their expenses. Would it be cheaper, in terms of the consumer compensation to cover them for a randomly fluctuating energy price, or an energy price which is pegged at its expected value?

Prefer random price: The government will be interested in how much it will cost them to keep consumers at a given utility level. This is just the expenditure function, $e(p, u)$. We know that this is concave in p. To really clarify this, think of all prices as fixed, and energy prices as fluctuating. From concavity, and Jensen's inequality we know that

$$E(e(p, u)) < e(E(p), u).$$

Two side comments on this: This is the usual machinery we used for expected utility, but a different application. Second, remember the basic intuition for why the expenditure function should be concave in prices. This is important. Draw it for one price, and think about why it needs to bend below a linear relationship.

3. In a multi good world, if two goods are consumed in equal quantities, $x_i = x_i$, and have equal income elasticities of demand, $\eta_i = \eta_i$, then they will exhibit symmetry in the *Marshallian* demands.

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_j(p, y)}{\partial p_i}$$

True: Although not true in general, this special case works. Write down the standard Slutsky equation:

$$\frac{\partial x_i(p, y)}{\partial p_j} = \frac{\partial x_i^h(p, y)}{\partial p_j} - x_j \frac{\partial x_i(p, y)}{\partial y}$$

$x_i = x_j$ and the condition on the demand elasticities gives

$$\frac{\partial x_i(p, y)}{\partial y} = \frac{\partial x_j(p, y)}{\partial y}$$

These facts along with symmetry in the Hicksian partials gives symmetry in the Marshallians. Remember, this is a VERY special case. Not true in general.

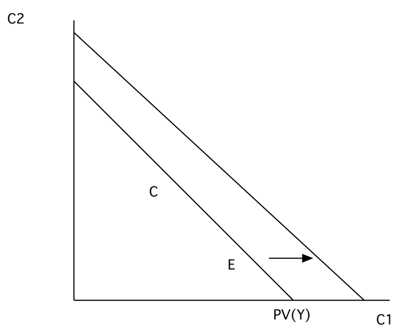
4. In a world with perfect capital markets where consumers may freely borrow and lend at a constant interest rate, r, they will chose life plans which maximize their present value of income regardless of their rate of time preference, β.

   This question yielded many interpretations, and if you were reasonable you should have received full credit. However, I was thinking about the following issue.

   Basically, this is true. Agents can separate income and consumption decisions in this world with perfect borrowing and lending. This is best shown in a picture of the two period world,

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} = PV(y)$$

   C is the agent's consumption, and E is the income endowment. Any strategy which boosts the present value of income would be a pure income increase in this world, and so everyone would want this regardless of preferences, or rate of time preference (really part of preferences).



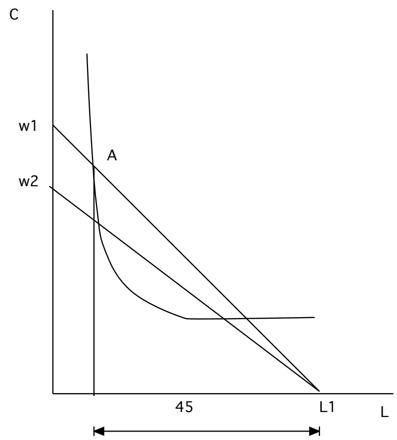
5. When a consumer is a net supplier of a good, and this good is inferior we cannot determine whether the own price elasticity of the good is positive or negative.

   False: Start with the Slutsky equation with net endowments.

$$\frac{\partial x_i(p,y)}{\partial p_i} = \frac{\partial x_i^h(p,y)}{\partial p_i} - (x_j - x_j^e)\frac{\partial x_i(p,y)}{\partial y}$$

   In this case $x_j - x_j^e$ is negative. Given an inferior good, that allows one to sign the right most term as (-)*(-) or (+). The own Hicks price impact (substitution) is also negative, so this gives a definite sign to the left side partial, and therefore also to the own elasticity of demand.

6. A firm offers a wage rate w, and forces workers to work a 45 hour week which is greater than their desired hours of work. A second firm could then enter the market and offer a contract which would attract these workers at a lower wage (lower labor costs).



True: In the figure the worker is forced to work 45 hours (point A). Therefore, they are not on an optimal labor/leisure choice. It is clear that another firm could offer a flexible hours contract at a lower wage which would move an agent to a higher indifference curve (would be preferred).

There are some caveats as to whether there are enough people in the economy, but I didn't really want you to go there. Some of you did bring this up.

II. Longer problems

1. (15 points) An investor is considering an investment in two risky assets with returns r_1 and r_2. These returns are independent and identically distributed. The investor is an expected utility maximizer with preferences given by

$$U = E(u(W(\alpha r_1 + (1 - \alpha) r_2)))$$

   (a) $u'(x) > 0$ and $u''(x) < 0$. Show that a portfolio with $\alpha = 0.5$ satisfies the first order condition for a maximum.

$$U = E(u(W(\alpha(r_1 - r_2) + r_2)))$$

$$dU/d\alpha = E(u'(W(\alpha(r_1 - r_2) + r_2))W(r_1 - r_2)) = 0$$

$$dU/d\alpha = E(u'(W(\alpha(r_1 - r_2) + r_2))(r_1 - r_2)) = 0$$

$$E(u'(W(\alpha(r_1 - r_2) + r_2))r_1) = E(u'(W(\alpha(r_1 - r_2) + r_2))r_2)$$

   At $\alpha = 1/2$,

$$E(u'(W(((1/2)r_1 + (1/2)r_2))r_1) = E(u'((1/2)r_1 + (1/2)r_2))r_2)$$

   Since r_1 and r_2 are iid, this is clearly satisfied, so $\alpha = 1/2$ satisfies the FOC.

   (b) Show that it also satisfies the 2nd order condition as well.

$$dU/d\alpha = E(u'(W(\alpha(r_1 - r_2) + r_2))W(r_1 - r_2))$$

$$d^2U/d\alpha^2 = E(u''(W(\alpha(r_1 - r_2) + r_2))W^2(r_1 - r_2)^2)$$

   Since $u''(x) < 0$ this is negative, and the second order condition is satisfied.

(c) Now assume that $u(x) = x^2$. What is the solution now for the utility maximizing value of α? If your answer has changed, explain why it changed.

Actually, the derivation from part (a) holds for this part too, so we know that $\alpha = 1/2$ satisfies this FOC.

However, the SOC is quite different, since $u''(x) = 2$,

$$d^2U/d\alpha^2 = E(2W^2(r_1 - r_2)^2)$$

This is clearly positive. $\alpha = 1/2$ is a local min, not a local max.

Also, this means that $u'(x)$ is increasing over all α. For $\alpha < 1/2$ it must be negative, and for $\alpha > 1/2$ it must be postive.

If alpha is bounded between $[0, 1]$ then it will be maxed at either 0 or 1 (from symmetry either one is ok).

At either one of these points the objective is still increasing, and if α were allowed to be anything the agent would continue to push it positive or negative, borrowing one asset and longing the other one.

All of this is due to the fact that the agent is risk loving in this case, and will increase risks as far as they can.

You can also solve directly for α in this case.

The FOC is

$$dU/d\alpha = E(u'(W(\alpha(r_1 - r_2) + r_2))W(r_1 - r_2))$$

$$dU/d\alpha = E(2W(\alpha(r_1 - r_2) + r_2))W(r_1 - r_2)) = 0$$

$$dU/d\alpha = E((\alpha(r_1 - r_2) + r_2))(r_1 - r_2)) = 0$$

$$E(\alpha(r_1 - r_2)^2 + r_2(r_1 - r_2))) = 0$$

$$E(\alpha(r_1^2 - 2r_1r_2 + r_2^2) + r_2r_1 - r_2^2))) = 0$$

Use iid here. They are the same.

$$\alpha E(2r_1^2 - 2r_1r_2) = E(r_1^2 - r_1r_2)$$

$$\alpha = 1/2$$

2. (10 points) In a world with identical homothetic preferences you know that the indirect utility function for all consumers i takes the form,

$$v(p, y) = g(p)y_i.$$

From this starting point show that the aggregate consumption of all consumers does not depend on the specific distribution of income, y_i.

Use Roy's identify to show that the demands are linear in y_i.

$$x_{j,i}(p, y) = \frac{\partial g(p)/\partial p_j}{g(p)} y_i$$

Summing this over all people i shows that demand for x_j depends only on aggregate income.

3. (13 points) An expected utility maximizing consumer has preferences given by

$$EU = E(g(x))$$

$$g(x) = \sqrt(x)$$

Consider a 3 outcome world, with payouts, $x_1, x_2, x_3 = (1, 4, 16)$, and probabilities given by $p_1, 1 - p_1 - p_3, p_3$. Now consider a small spreading out of the probabilities from event 2, to events 1 and 3. It is given by,

$$(p_1 + 2a, 1 - (p_1 + 2a) - (p_3 + a), p_3 + a).$$

(a) Find dEU/da.

$$EU = p_1 g(x_1) + p_2 g(x_2) + p_3 g(x_3)$$

$$EU = p_1(1) + p_2(2) + p_3(4)$$

$$EU = (p_1 + 2a)(1) + (1 - (p_1 + 2a) - (p_3 + a)(2) + (p_3 + a)(4)$$

$$dEU/da = E(dU/da) = 2(1) - 3(2) + (4) = 0$$

This is an indifference curve in probability/utility space (Machina).

(b) Find dE(x)/da.

$$Ex = (p_1 + 2a)(1) + (1 - (p_1 + 2a) - (p_3 + a)(4) + (p_3 + a)(16)$$

$$dE(x)/da = 2(1) - 3(4) + (16) = 6$$

(c) It is generally thought that a spreading out of probabilities is considered to be adding risk. Why is this consistent with your answers to parts (a) and (b)?

From the concavity of the objective function this person is risk averse. Probabilities spread out (risk increases), and the expected value of the gamble increases. However, the utility stay the same. This shows that the increase in risk is exactly compensating for the increase in expected value, which is consistent with a risk averse agent.

One could also say that moving on an iso-utility curve moves one upward in expected value terms, which is the graphical description of the same thing.

4. (20 points) Consider an intertemporal expected utility optimizing consumer with preferences given by

$$U = \sum_{t=0}^{5} \beta^t (\frac{-1}{c_t})$$

Consumers receive a lump sum of income today, y, and have access to borrowing and lending markets at an interest rate of r per period. Therefore, their budget constraint looks like,

$$y = \sum_{t=0}^{5} \frac{c_t}{(1+r)^t}$$

(a) Solve the usual Lagrangian for this consumer to get the FOC's for consumption each period.

$$L = \sum_{t=0}^{5} \beta^t (\frac{-1}{c_t}) + \lambda(y - \sum_{t=0}^{5} \frac{c_t}{(1+r)^t})$$

$$\frac{\partial L}{\partial c_t} = \beta^t \frac{1}{c_t^2} - \lambda \frac{1}{(1+r)^t} = 0$$

$$\frac{\partial L}{\partial \lambda} = y - \sum_{t=0}^{5} \frac{c_t}{(1+r)^t} = 0$$

(b) Is there a simple condition on the values of β and $(1+r)$ that could tell you whether consumption was increasing or decreasing over time?

From the FOC's you get the following relationship between c_t and c_{t+1}.

$$(1/\beta)\frac{c_{t+1}^2}{c_t^2} = (1+r)$$

$$\frac{c_{t+1}}{c_t} = \sqrt{\beta(1+r)} = \gamma$$

Obviously, the relative values of β and r control the growth rate of consumption. Impatient consumers, $\beta(1+r) < 1$ gives consumption falling. $\beta(1+r) > 1$ gives a rising time path for consumption.

(c) Find the optimal level of consumption for the consumer today, c_0.

From the above expression,

$$c_t = \gamma^t c_0$$

Now sub in the budget constraint,

$$y = \sum_{t=0}^{5} \frac{\gamma^t c_0}{(1+r)^t}$$

$$c_0 = \frac{y}{(\sum_{t=0}^{5} \frac{\gamma^t}{(1+r)^t})}$$

(d) Show that the expenditure function is linear in -1/u. In other words it is

$$e(r, u; \beta) = g(r; \beta)(-1/u)$$

$$c_0 = \frac{y}{(\sum_{t=0}^{5} \frac{\gamma^t}{(1+r)^t})}$$

$$c_0 = yh(r, \beta)$$

$$c_t = yh(r, \beta)\gamma^t$$

Indirect utility form:

$$U = \sum_{t=0}^{5} \beta^t (\frac{-1}{yh(r, \beta)\gamma^t})$$

$$U = (-1/y)\sum_{t=0}^{5}\beta^t(\frac{1}{h(r,\beta)\gamma^t})$$

Now invert this to get the form of the expenditure

$$e = y = (-1/U)\sum_{t=0}^{5}\beta^t(\frac{1}{h(r,\beta)\gamma^t})$$

$$e = y = (-1/U)\sum_{t=0}^{5}\beta^t(\frac{1}{h(r,\beta)\gamma^t}) = (-1/U)g(r;\beta)$$