Problem Set #2. Econ 301 Fall 2010.

1. Define $S_{ij} = \frac{\partial x_i(p,y)}{\partial p_j} + \frac{\partial x_i(p,y)}{\partial y} \cdot x_j^0 = \frac{\partial x_i^c(p, px^0)}{\partial p_j}$

$$\underset{L\times L}{S} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_L \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1L} \\ S_{21} & S_{22} & \cdots & S_{2L} \\ \cdots \\ S_{L1} & S_{L2} & \cdots & S_{LL} \end{bmatrix}, \qquad p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix}$$

$$S \cdot p = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1L} \\ S_{21} & S_{22} & \cdots & S_{2L} \\ \cdots \\ S_{L1} & S_{L2} & \cdots & S_{LL} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{L} S_{1j} \cdot p_j \\ \sum_{j=1}^{L} S_{2j} \cdot p_j \\ \cdots \\ \sum_{j=1}^{L} S_{Lj} \cdot p_j \end{bmatrix}$$

i-th element of $S \cdot p$ is:

$$\sum_{j=1}^{L} S_{ij} \cdot p_j = \sum_{j=1}^{L} \frac{\partial x_i^c(p, px^0)}{\partial p_j} \cdot p_j$$

$\because$ $x_i^c(p, px^0)$ is homogeneous of degree zero in p

$\therefore$ By Euler's theorem, we have:

$$\sum_{j=1}^{L} \frac{\partial x_i^c(p, px^0)}{\partial p_j} \cdot p_j = 0$$

Therefore, $\sum_{j=1}^{L} S_{ij} \cdot p_j = 0$, $\quad S \cdot p = 0$

$$p \cdot S = [p_1\ p_2 \cdots p_L] \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_L \end{bmatrix} = p_1 S_1 + p_2 S_2 + \cdots + p_L S_L = \begin{bmatrix} \sum_{i=1}^{L} p_i S_{i1} \\ \sum_{i=1}^{L} p_i \cdot S_{i2} \\ \cdots \\ \sum_{i=1}^{L} p_i \cdot S_{iL} \end{bmatrix}$$

j-th element of $p \cdot S$ is:

$$\sum_{i=1}^{L} p_i \cdot S_{ij} = \sum_{i=1}^{L} p_i \cdot \frac{\partial x_i^c(p, px)}{\partial p_j} = \sum_{i=1}^{L} p_i \left(\frac{\partial x_i(p,y)}{\partial p_j} + \frac{\partial x_i(p,y)}{\partial y} \cdot x_j \right)$$

$$\therefore \sum_{i=1}^{L} p_i \cdot S_{ij} = \sum_{i=1}^{L} p_i \frac{\partial x_i(p,y)}{\partial p_j} + \sum_{i=1}^{L} p_i \cdot \frac{\partial x_i(p,y)}{\partial y} \cdot x_j$$

$$= \sum_{i=1}^{L} p_i \cdot \frac{\partial x_i(p,y)}{\partial p_j} + x_j \sum_{i=1}^{L} p_i \cdot \frac{\partial x_i(p,y)}{\partial y} \quad \cdots (1)$$

By Walras' law, we have:

$$\sum_{i=1}^{L} p_i \, x_i(p,y) = y \quad \cdots (2)$$

Differentiate (2) with respect to p_j:

$$x_j(p,y) + \sum_{i=1}^{L} p_i \frac{\partial x_i(p,y)}{\partial p_j} = 0 \quad \cdots (3)$$

Differentiate (2) with respect to y:

$$\sum_{i=1}^{L} p_i \frac{\partial x_i(p,y)}{\partial y} = 1 \quad \cdots (4)$$

Substitute (3) & (4) into (1):

$$\sum_{i=1}^{L} p_i \cdot S_{ij} = -x_i(p,y) + x_i(p,y) \cdot 1 = 0$$

Therefore, $\sum_{i=1}^{L} p_i \cdot S_{ij} = 0, \ \forall j$, $p \cdot S = 0$

2. Assume "$\succsim$" is complete and transitive.

(i). If $x \succ y$, $y \succ z$

By definition of "$\succ$", we have:

$x \succsim y$, $y \not\succsim x$

$y \succsim z$, $z \not\succsim y$

$\because$ "$\succsim$" is transitive

$\therefore x \succsim y, y \succsim z \Rightarrow x \succsim z$ ①

Suppose $z \succsim x$

$\because$ $x \succsim y$

$\therefore$ By transitivity of "$\succsim$", we have $z \succsim x \succsim y$, $z \succsim y$

$\because$ "$z \succsim y$" contradicts to "$z \not\succsim y$"

$\therefore$ $z \not\succsim x$ ②

Combine ① & ②, we get $x \succ z$

(ii). If $x \sim y$, $y \sim z$

By definition of "$\sim$", we have:

$x \succsim y$, $y \succsim x$

$y \succsim z$, $z \succsim y$

$\because$ "$\succsim$" is transitive

$\therefore$ $x \succsim y$, $y \succsim z \Rightarrow x \succsim z$ ③

$z \succsim y$, $y \succsim x \Rightarrow z \succsim x$ ④

Combine ③ & ④, we get $x \sim z$.

3. • Axiom 4': For all $x^0 \in R^n_+$, and for all $\varepsilon > 0$, where exists some $x \in B_\varepsilon(x^0) \cap R^n_+$, such that $x \succ x^0$

• Axiom 4: For all $x^1, x^0 \in R^n_+$, if $x^0 \geq x^1$, then $x^0 \succsim x^1$, while if $x^0 >> x^1$, then $x^0 \succ x^1$.

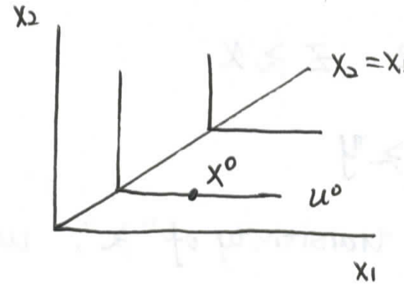
Graphically, for any $x^0 \in R^n_+$,

Axiom 4' says $x \succ x^0$ could be some point in $B_\varepsilon(x^0)$.

$B_\varepsilon(x^0)$

Axiom 4 says $x \succ x^0$ could be only the points noreheast to x^0. clearly, Axiom 4 is more restrictive than Axiom 4'.

4. Assume x° is a point on indifference curve u°.



(i). By Axiom 4', we can alway find a point

$x' \in B_\varepsilon(x^\circ)$, such that $x' > x^\circ$

$\because$ Preference increases northeasterly

$\therefore$ x' must be northeastern to x°.

$\therefore$ All linear combinations of x' and x°, i.e $tx^\circ + (1-t)x'$ for $\forall t \in [0,1]$

are either northeastern to x° or vanish to x°

$\therefore$ $tx^\circ + (1-t)x' \succsim x^\circ$ for $\forall t \in [0,1]$

If $x' \sim x^\circ$,

then x' lies on indifference curve u° as well.

$\therefore$ $tx^\circ + (1-t)x'$ for $\forall t \in [0,1]$ are either above u° or on u°

$\therefore$ $tx^\circ + (1-t)x' \succsim x^\circ$ for $\forall t \in [0,1]$

Therefore, if $x' \succsim x^\circ$, then $tx^\circ + (1-t)x' \succsim x^\circ$ for $\forall t \in [0,1]$

Axiom 5' satisfies.

(ii). Let's simplify to 2-dimensional case.

$x^\circ = (x_1^\circ, x_2^\circ)$, $x' = (x_1', x_2')$

If $x' >> x^\circ$, equivalent to $x_1' > x_1^\circ$ and $x_2' > x_2^\circ$

then x' is northeastern to x°

then $x' > x^\circ$ under the assumed preference.

If $x^1 \geq x^0$,

then it must be one of following 4 cases:

$$\left.\begin{array}{ll} x_1^1 = x_1^0,\ x_2^1 = x_2^0 & \Rightarrow x^1 \sim x^0 \\ x_1^1 = x_1^0,\ x_2^1 > x_2^0 & \Rightarrow x^1 \gtrsim x^0 \\ x_1^1 > x_1^0,\ x_2^1 = x_2^0 & \Rightarrow x^1 \gtrsim x^0 \\ x_1^1 > x_1^0,\ x_2^1 > x_2^0 & \Rightarrow x^1 \succ x^0 \end{array}\right\} \Rightarrow x^1 \gtrsim x^0$$

Therefore, if $x^1 \geq x^0$, then $x^1 \gtrsim x^0$; if $x^1 \gg x^0$, then $x^1 \succ x^0$

Axiom 4 satisfies.

(iii). Let $x^1 = (x_1^1, x_2^1)$, such that $x_1^1 > x_1^0$ and $x_2^1 = x_2^0$

Under the assume preference, $x^1 \sim x^0$

Obviously, $x^1 \neq x^0$, and $x^1 \gtrsim x^0$

However, $tx^0 + (1-t)x^1 \sim x^0$ for $\forall t \in [0,1]$

Therefore, Axiom 5 does not satisfy.

(iv). In J&R, $x_1^1 > x_1^0$ and $x_2^1 > x_2^0 \Rightarrow x^1 \succ x^0$

In Varian,

$$\left.\begin{array}{l} x_1^1 > x_1^0 \text{ and } x_2^1 = x_2^0 \\ x_1^1 = x_1^0 \text{ and } x_2^1 > x_2^0 \\ x_1^1 > x_1^0 \text{ and } x_2^1 > x_2^0 \end{array}\right\} \Rightarrow x^1 \succ x^0$$

clearly, J&R's is a stronger assumption than Varian's.

5. $A \Rightarrow B$. B is a necessary condition to A.

In this case,

A = existence of a continuous utility function $u(x)$
B = Axiom 3 holds

$A \Rightarrow B$ is equivalent to $B^c \Rightarrow A^c$

So my strategy is to prove: if Axiom 3 fails, then $u(x)$ does not exist.

If Axiom 3 fails,

then $\succsim(x^0)$ is not closed.

then we can find at least one sequence $\{x^k\}_{k=1}^{\infty}$ in $\succsim(x^0)$,

such that $\lim\limits_{k\to\infty} x^k = x$, $x \notin \succsim(x^0)$

then $\lim\limits_{k\to\infty} x^k = x$, $x \in \prec(x^0)$ or $x \prec x^0$

Suppose there exists a continuous utility function $u(x)$,

$\because$ $x \prec x^0$

$\therefore$ $u(x) < u(x^0)$

$\because$ $\{x^k\}_{k=1}^{\infty} \in \succsim(x^0)$

$\therefore$ $\{u(x^k)\}_{k=1}^{\infty} \geqslant u(x^0)$

$\because$ $u(x)$ is continuous

$\therefore$ $\lim\limits_{k\to\infty} u(x^k) = u(\lim\limits_{k\to\infty} x^k) = u(x)$

By the property of limit, we have:

$u(x) = \lim\limits_{k\to\infty} u(x^k) \geq u(x^0)$

This contradicts to $u(x) < u(x^0)$

Therefore, continuous utility function $u(x)$ does not exist.