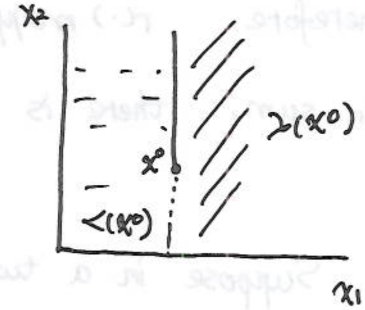
PS #3   Econ 301   Fall 2010

1. For lexicographic preference,



$\sim(x^0)$ degenerate to the point itself x^0.

$\succ(x^0)$ & $\prec(x^0)$ are shown in the right picture.

①. suppose in the two dimensional space, there are two points:

$x_n = (1+\frac{1}{n}, 1)$, $y_n = (1, 2)$

According to lexicographic preference,

$\because$ $1+\frac{1}{n} > 1$

$\therefore$ $x_n \succ y_n$

$\because$ as $n \to \infty$, $x_n \to (1, 1)$, $y_n \to (1, 2)$

apparently, $1 < 2$

$\therefore$ $x_n \prec y_n$, which violates the continuity.

②. suppose there is a continuous utility function $u(\cdot)$, representing lexicographic preference.

For every x_1, $x_1 \in [0, +\infty)$, we can pick up a rational number $r(x_1)$, such that $u(x_1, 2) > r(x_1) > u(x_1, 1)$.

Under lexicographic preference, we have $r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1')$ if $x_1 > x_1'$. Hence, if $x_1 > x_1'$, then $r(x_1) > r(x_1')$.

This means $r(\cdot)$ provides a one-to-one mapping from the set of real numbers to the set of rational numbers.

However, the two sets have different cardinals, real numbers are uncountable, rational numbers are countable.

Therefore, $r(\cdot)$ mapping is mathematically impossible.

In sum, there is no utility function for lexicographic preference.

2. Suppose in a two-good world, consumers' problem is:

$$\max_{x_1, x_2} u(x_1, x_2)$$

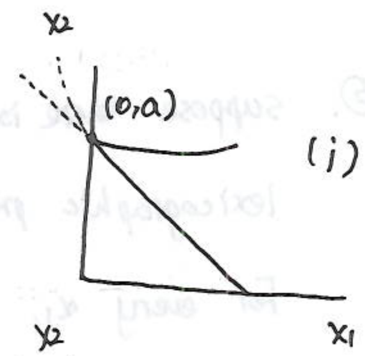
$$s.t.: \ p_1 x_1 + p_2 x_2 \le M$$

$$L(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2)$$

ⓐ. If the optimal choice is an interior solution,

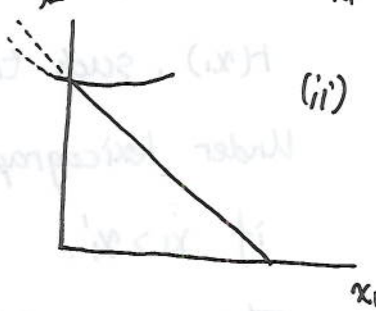
then
$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= u_1(x_1, x_2) - \lambda p_1 = 0 \\ \frac{\partial L}{\partial x_2} &= u_2(x_1, x_2) - \lambda p_2 = 0 \end{aligned}\right\} \Rightarrow \frac{p_1}{p_2} = \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = \frac{dx_2}{dx_1}$$
$$\frac{\partial L}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 = 0$$

ⓑ. If the optimal choice is at corner, say $(0, a)$, it could be one of the following 2 cases:



(i). the indifference curve is tangential to the budget constraint at $(0, a)$

so $\frac{\partial L}{\partial x_1} = 0$ and $\frac{\partial L}{\partial x_2} = 0$ still hold.



(ii). the indifference curve is not tangential to the budget constraint at $(0, a)$, as depicted in the right diagram.

so $\frac{dx_2}{dx_1} = \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} < \frac{p_1}{p_2}$

then $\frac{\partial L}{\partial x_1} < 0, \quad x_1 = 0 \longleftarrow$ K-T condition

$\frac{\partial L}{\partial x_2} = 0, \quad x_2 > 0$

Combine (i) & (ii), we have the following conditions for corner solutions:

$$\frac{\partial L}{\partial x_1} \le 0, \qquad x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 0, \qquad x_2 > 0$$

3. $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$

(a). $u_1 = \frac{1}{3} x_1^{-\frac{2}{3}} x_2^{\frac{1}{3}}$, $u_2 = \frac{1}{3} \cdot x_1^{\frac{1}{3}} \cdot x_2^{-\frac{2}{3}}$

$u_{11} = -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{1}{3}}$, $u_{12} = u_{21} = \frac{1}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{2}{3}}$, $u_{22} = -\frac{2}{9} x_1^{\frac{1}{3}} x_2^{-\frac{5}{3}}$

$$\rightarrow H = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{1}{3}} & \frac{1}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{2}{3}} \\ \frac{1}{9} x_1^{-\frac{2}{3}} x_2^{-\frac{2}{3}} & -\frac{2}{9} x_1^{\frac{1}{3}} x_2^{-\frac{5}{3}} \end{bmatrix}$$

→ Principal minors of H:

$D_1 = u_{11} = -\frac{2}{9} x_1^{-\frac{5}{3}} x_2^{\frac{1}{3}} < 0$ for all $x_1 > 0$, $x_2 > 0$

$D_2 = |H| = \frac{4}{81} x_1^{-\frac{4}{3}} x_2^{-\frac{4}{3}} - \frac{1}{81} x_1^{-\frac{4}{3}} x_2^{-\frac{4}{3}} = \frac{3}{81} x_1^{-\frac{4}{3}} x_2^{-\frac{4}{3}} > 0$ for all $x_1 > 0, x_2 > 0$

→ H is negative positive

→ $u(x_1, x_2)$ is strictly concave.

(b). $V(x_1, x_2) = [u(x_1, x_2)]^6 = x_1^2 \cdot x_2^2$

$V_1 = 2x_1 x_2^2$, $V_2 = 2x_1^2 x_2$

$V_{11} = 2x_2^2$, $V_{12} = V_{21} = 4x_1 x_2$, $V_{22} = 2x_1^2$

$$\rightarrow H = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} 2x_2^2 & 4x_1 x_2 \\ 4x_1 x_2 & 2x_1^2 \end{bmatrix}$$

→ Principal minors of H:

$D_1 = 2x_2^2 \geq 0$ for all $x_1>0,\ x_2>0$

$D_2 = |H| \doteq 4x_1^2x_2^2 - 16x_1^2x_2^2 = -12x_1^2x_2^2 \leq 0$ for all $x_1>0,\ x_2>0$

→ H is indefinite

→ $v(x_1, x_2)$ is neither concave nor convex.

(c). If $u(x_1,x_2)$ is quasi-concave,

then $\succsim(x^0)$ are convex for all x^0, or indifference curves are convex to the origin.

Because monotonic transformation of $u(x_1,x_2)$ does not affect the curvature of indifference curves, or $\succsim(x^0)$ are still convex under $V(x_1,x_2) = F[u(x_1,x_2)]$, $F'(\cdot)>0$

So $v(x_1,x_2)$ is quasi-concave as well.

In short, if $u(\vec{x})$ is quasi-concave, then $F[u(\vec{x})]$, $F'(\cdot)>0$ is quasi-concave as well.

- For $u(x_1,x_2)$, slope of indifference level is: $\frac{dx_2}{dx_1} = -\frac{u_1(x_1,x_2)}{u_2(x_1,x_2)}$

For $v(x_1,x_2) = F[u(x_1,x_2)]$, slope of indifference curve is:

$$\frac{dx_2}{dx_1} = -\frac{V_1(x_1,x_2)}{V_2(x_1,x_2)} = -\frac{F'(u)\cdot u_1(x_1,x_2)}{F'(u)\cdot u_2(x_1,x_2)} = -\frac{u_1(x_1,x_2)}{u_2(x_1,x_2)}$$

Hence, monotonic transformation of $u(x_1,x_2)$ does not affect the slope of the indifference curve.

4. Consumers' problem:

$$\max_{\vec{x}} u(\vec{x}) \quad s.t.: \ \vec{p}\cdot\vec{x} \le y \quad \Rightarrow L(\vec{x},\lambda) = u(\vec{x}) + \lambda(y - \vec{p}\vec{x})$$

By Roy's identity,

$$x_j(\vec{p}, y) = -\frac{\partial L/\partial p_j}{\partial L/\partial y}, \qquad x_i(p,y) = -\frac{\partial L/\partial p_i}{\partial L/\partial y}$$

$$\therefore \frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial}{\partial p_j}\left[-\frac{\partial L/\partial p_i}{\partial L/\partial y}\right] = -\frac{1}{\left(\frac{\partial L}{\partial y}\right)^2}\left[\frac{\partial^2 L}{\partial p_i \partial p_j}\cdot\frac{\partial L}{\partial y} - \frac{\partial L}{\partial p_i}\cdot\frac{\partial^2 L}{\partial y \partial p_j}\right]$$

$$\frac{\partial x_j(p,y)}{\partial p_i} = -\frac{1}{\left(\frac{\partial L}{\partial y}\right)^2}\left[\frac{\partial^2 L}{\partial p_j \partial p_i}\cdot\frac{\partial L}{\partial y} - \frac{\partial L}{\partial p_j}\cdot\frac{\partial^2 L}{\partial y \partial p_i}\right]$$

$$\because \quad x_i = x_j$$

$$\therefore \quad \left.\begin{array}{l} \frac{\partial L}{\partial p_i} = -\lambda x_i \\ \frac{\partial L}{\partial p_j} = -\lambda x_j \end{array}\right\} \Rightarrow \frac{\partial L}{\partial p_i} = \frac{\partial L}{\partial p_j}$$

$$\because \quad \eta_i = \eta_j, \quad x_i = x_j$$

$$\therefore \quad \frac{\partial x_i}{\partial y}\cdot\frac{y}{x_i} = \frac{\partial x_j}{\partial y}\cdot\frac{y}{x_j}$$

$$\therefore \quad \frac{\partial x_i}{\partial y} = \frac{\partial x_j}{\partial y}$$

$$\because \quad \frac{\partial^2 L}{\partial y \partial p_j} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial p_j}\right) = \frac{\partial}{\partial y}(-\lambda x_j) = -\left(\frac{\partial \lambda}{\partial y}\cdot x_j + \lambda\cdot\frac{\partial x_j}{\partial y}\right)$$

$$\frac{\partial^2 L}{\partial y \partial p_i} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial p_i}\right) = \frac{\partial}{\partial y}(-\lambda x_i) = -\left(\frac{\partial \lambda}{\partial y}\cdot x_i + \lambda\cdot\frac{\partial x_i}{\partial y}\right)$$

$$\therefore \quad \frac{\partial^2 L}{\partial y \partial p_j} = \frac{\partial^2 L}{\partial y \partial p_i}, \text{ since } x_i = x_j, \ \frac{\partial x_j}{\partial y} = \frac{\partial x_i}{\partial y}$$

$$\therefore \quad \frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial x_j(p,y)}{\partial p_i}$$

5. $$\max_{\vec{x}} u(\vec{x}) = \sum_{i=1}^{n} u_i(x_i)$$

$$\text{s.t.} \quad \sum p_i x_i \le y$$

$$\mathcal{L}(x_1, \cdots x_n; \lambda) = \sum_{i=1}^{n} u_i(x_i) + \lambda\left(y - \sum_{i=1}^{n} p_i x_i\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = u_i'(x_i) - \lambda p_i = 0 \qquad i = 1, 2, \cdots n$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - \sum_{i=1}^{n} p_i x_i = 0$$

It is true that $MRS = \frac{dx_i}{dx_j} = -\frac{u_j'(x_j)}{u_i'(x_i)} = -\frac{p_j}{p_i}$ by FOCs.

however, from $u_i'(x_i) - \lambda p_i = 0$ we can see x_i depends on p_i and λ, which in turn is a function of $(p_1, \cdots p_n; y)$

Therefore, x_i does not necessarily depend only on p_i & p_j.

↳ Example 1: consider $u(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

demand functions are: $x_i = \frac{p_i \cdot y}{p_1^2 + p_2^2 + p_3^2}$

Example 2: consider $u(x_1, x_2, x_3) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \alpha_3 \ln x_3$

demand functions are: $x_i = \frac{\alpha_i}{\alpha_1 + \alpha_2 + \alpha_3} \cdot \frac{y}{p_i}$