P set #4 Econ 301 Fall 2010

1. (a). Let p_1 = price of food; p_2 = price of other goods; y = disposable income

x_1 = amount of food; x_2 = amount of other goods

Under initial prices and income (p_1^0, p_2^0, y^0), the initial choices are (x_1^0, x_2^0), such that $p_1^0 \cdot x_1^0 + p_2^0 \cdot x_2^0 = y^0$

Under new prices and income $(1.1p_1^0, p_2^0, 1.05y^0)$,

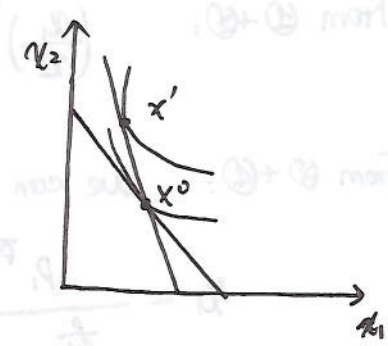
$$x_1^0 \cdot p_1' + x_2^0 \cdot p_2' = x_1^0 * 1.1 p_1^0 + x_2^0 * p_2^0$$
$$= 1.1\, x_1^0 p_1^0 + x_2^0 \cdot p_2^0 \quad \because x_1^0 p_1^0 = \tfrac{1}{2} y^0$$
$$= 1.1 * 0.5 y^0 + \tfrac{1}{2} y^0$$
$$= 1.05 y^0$$
$$= y'$$

$\therefore \quad x_1^0 \cdot p_1' + x_2^0 \cdot p_2' = y'$

(x_1^0, x_2^0) are still feasible under $(1.1p_1^0, p_2^0, 1.05y^0)$

$\therefore$ the new budget rotates around (x_1^0, x_2^0)

$\therefore$ the consumer will be better off, he will consume X', reaching a higher utility level.



(b). $\because$ indirect utility function is homogeneous of degree 0 in $(p_1 \cdots p_n, y)$

$$\therefore \quad \frac{g(tp_1, \cdots tp_n)}{ty} + h(tp_1, \cdots tp_n) = v(tp_1, \cdots tp_n, ty)$$
$$= v(p_1, \cdots p_n, y)$$
$$= \frac{g(p_1 \cdots p_n)}{y} + h(p_1 \cdots p_n) \qquad (*)$$

$\therefore\ g(tp_1, \cdots tp_n) = t g(p_1, \cdots p_n)$, $h(tp_1 \cdots tp_n) = h(p_1, \cdots p_n)$ to make sure (*) holding

$\therefore\ g(p_1, \cdots p_n)$ must be homogeneous of degree 1, $h(p_1, \cdots p_n)$ must be homogeneous of degree 0.

2. (a). $v(x_1, x_2) = \rho \log u(x_1, x_2) = \rho \operatorname{Log}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = \operatorname{Log}(x_1^\rho + x_2^\rho)$

$\because\ \rho \log(u)$ is monotonically increasing,

$\therefore\ v(x_1, x_2)$ represent the same preference as $u(x_1, x_2)$ does.

(b). $L(x_1, x_2, \lambda) = \log(x_1^\rho + x_2^\rho) + \lambda(y - p_1 x_1 - p_2 x_2)$

FOCs:

$$\frac{\partial L}{\partial x_1} = \frac{\rho \cdot x_1^{\rho-1}}{x_1^\rho + x_2^\rho} - \lambda p_1 = 0 \qquad ①$$

$$\frac{\partial L}{\partial x_2} = \frac{\rho \cdot x_2^{\rho-1}}{x_1^\rho + x_2^\rho} - \lambda p_2 = 0 \qquad ②$$

$$\frac{\partial L}{\partial \lambda} = y - p_1 x_1 - p_2 x_2 = 0 \qquad ③$$

From ①+②: $\left(\frac{x_1}{x_2}\right)^{\rho-1} = \frac{p_1}{p_2} \Rightarrow x_1 = x_2 \cdot \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$ ④

From ③+④: we can get the Marshallian demands

$$x_1 = \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \cdot y$$

$$x_2 = \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \cdot y$$

(c). $$x_i(tp_1, tp_2, ty) = \frac{(tp_i)^{\frac{1}{\rho-1}} \cdot ty}{(tp_1)^{\frac{\rho}{\rho-1}} + (tp_2)^{\frac{\rho}{\rho-1}}} = \frac{t^{\frac{\rho}{\rho-1}} \cdot p_i y}{t^{\frac{\rho}{\rho-1}}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)} = x_i(p_1, p_2, y) \quad i=1,2$$

$\therefore\ x_1(p_1, p_2, y)$, $x_2(p_1, p_2, y)$ are homogeneous of degree 1 in (p_1, p_2, y)

3. Under $u(x_1, x_2)$, $MRS = \frac{dx_2}{dx_1} = -\frac{u_1(x_1, x_2)}{u_2(x_1, x_2)} = -\frac{\alpha \cdot x_1^{\alpha-1} \cdot x_2^{1-\alpha}}{(1-\alpha)\, x_1^{\alpha} \cdot x_2^{-\alpha}} = -\frac{\alpha}{1-\alpha} \cdot \frac{x_2}{x_1}$

Under $v(x_1, x_2)$, $MRS = \frac{dx_2}{dx_1} = -\frac{v_1(x_1, x_2)}{v_2(x_1, x_2)} = -\frac{\gamma \cdot x_1^{\gamma-1} \cdot x_2^{\beta}}{\beta \cdot x_1^{\gamma} \cdot x_2^{\beta-1}} = -\frac{\gamma}{\beta} \cdot \frac{x_2}{x_1}$

$$= -\frac{\gamma/(\gamma+\beta)}{\beta/(\gamma+\beta)} \cdot \frac{x_2}{x_1}$$

Let $\alpha = \frac{\gamma}{\gamma+\beta}$, $1-\alpha = \frac{\beta}{\gamma+\beta}$

then MRS are equal under $u(x_1, x_2)$ and $v(x_1, x_2)$

$\because$ behaviors are characterized by MRS

$\therefore$ $u(x_1, x_2)$ & $v(x_1, x_2)$ represent the same preference.

4. (a). Yes. Because monotonic transformation, i.e $f'(\cdot) > 0$, does not change the preference. $v(x)$ & $u(x)$ describe the same behavior.

(b). Expenditure minimization:

$$\min_{\vec{x}} \ \vec{p} \cdot \vec{x}$$

$$\text{s.t.:} \quad g(\vec{x}) \geq u$$

$$L(\vec{x}, \lambda) = \vec{p} \cdot \vec{x} + \lambda(u - g(\vec{x}))$$

FOCs: $\frac{\partial L}{\partial x_i} = p_i - \lambda \cdot g_i(\vec{x}) = 0 \quad i=1,2\cdots n$ ①

$\frac{\partial L}{\partial \lambda} = u - g(\vec{x}) = 0$ ②

x_i, $i=1,2\cdots n$ are the solutions of the equation system ①-②.

$\because$ $g(t\vec{x}) = t\,g(\vec{x})$

$\therefore$ $g_i(t\vec{x}) = g_i(\vec{x})$

① + ② are equivalent to:

$p_i - \lambda \cdot g_i(t\vec{x}) = 0 \quad i=1,2\cdots n$ ③

$tu - g(t\vec{x}) = 0$ ④

③ + ④ means $tx_i, i=1,2\cdots n$ are the solutions of the equation system ③+④, corresponding to $(\vec{p}, tu)$

Hence, $t x_i(\vec{p}, u) = x_i(\vec{p}, tu)$

$$\therefore \quad e(\vec{p}, tu) \equiv \sum p_i \cdot x_i(\vec{p}, tu) \equiv \sum p_i \cdot t x_i(\vec{p}, u) \equiv t \sum p_i x_i(\vec{p}, u) \equiv t \cdot e(p, u)$$

Differentiate $e(\vec{p}, tu) \equiv t\, e(p, u)$ w.r.t t:

$$\frac{\partial e(\vec{p}, tu)}{\partial (tu)} \cdot u \equiv e(p, u) \quad \forall t$$

Let $t=1$, then

$$\frac{\partial e(\vec{p}, u)}{\partial u} \equiv \frac{e(p, u)}{u}$$

$$\int \frac{\partial e(\vec{p}, u)}{\partial u} \equiv \int \frac{e(p, u)}{u}$$

$\therefore \ln e(\vec{p}, u) = \ln u + \ln C_0$

$\therefore e(\vec{p}, u) = u \cdot C_0$

Let $u=1$, then $C_0 = e(\vec{p}, 1)$

$\therefore e(\vec{p}, u) = u \cdot e(\vec{p}, 1)$

Define $e(\vec{p}, 1) = e(\vec{p})$,

then $e(\vec{p}, u) = e(\vec{p})\, u$

(c). Similarly to (b), we could prove:

$$x_i(\vec{p}, ty) = t x_i(\vec{p}, y)$$

$$\to \quad V(\vec{p}, ty) = g[\vec{x}(\vec{p}, ty)] = g[t\vec{x}(\vec{p}, y)] = t g[\vec{x}(\vec{p}, y)] = t\, v(\vec{p}, y)$$

$$\to \quad V(\vec{p}, y) = v(p, 1) \cdot y = v(p) \cdot y \quad \text{where } v(p) = v(p, 1)$$

(d). By Roy's identy,

$$x_i(\vec{p}, y) = -\frac{\partial v(\vec{p}, y) / \partial p_i}{\partial v(\vec{p}, y) / \partial y} = -\frac{\partial v(p) / \partial p_i \cdot y}{v(p)} = -\frac{v_i(p)}{v(p)} \cdot y$$

Let $x_i(p) = -\frac{v_i(p)}{v(p)}$

then $x_i(\vec{p}, y) = x_i(p) \cdot y$

(e). $$\eta_i = \frac{\partial x_i(\vec{p}, y)}{\partial y} \cdot \frac{y}{x_i(p, y)} = x_i(p) \cdot \frac{y}{x_i(p, y)} = \frac{x_i(p, y)}{x_i(p, y)} = 1$$

$\therefore$ η_i is always equal to 1 regardless of y.