Problem Set 9. Econ 301.

1. (a). False. IRS does not imply homotheticity

For example, $f(x_1, x_2) = x_1 + x_2^2$

$$f(2x_1, 2x_2) = 2x_1 + 2x_2^2 + 2x_2^2 > 2x_1 + 2x_2^2$$

$$\Rightarrow f(2x_1, 2x_2) > 2f(x_1, x_2)$$

So $f(x_1, x_2)$ is IRS but not homothetic.

(b). True. CRS production functions are homogeneous of degree 1.

(c). False. Homotheticity in production does not control scale effects, only factor ~~product~~ proportions.

(d). True. $F(x) = G[f(x)]$ is a homothetic when $f(x)$ is homogeneous of degree 1 and $G(\cdot)$ is a strictly increasing function.

FOC in cost-minimization:

$$\frac{w_i}{w_j} = \frac{\partial F / \partial x_i}{\partial F / \partial x_j} = \frac{G' \cdot f_i(x)}{G' \cdot f_j(x)} = \frac{f_i(x)}{f_j(x)}$$

$\because$ $f(x)$ is homogeneous of degree 1

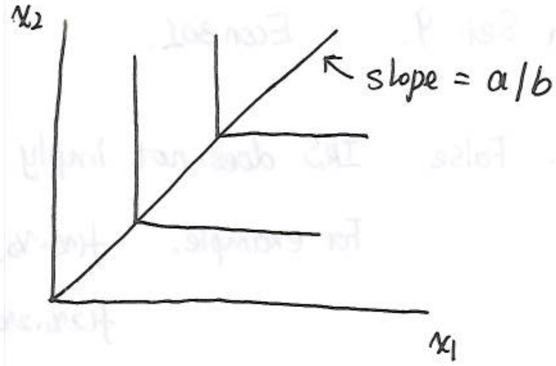
$\therefore$ $f_i(x)$ is homogeneous of degree 0

$$\therefore \frac{w_i}{w_j} = \frac{f_i(x)}{f_j(x)} = \frac{f_i(\lambda x)}{f_j(\lambda x)}$$

$\therefore$ As long as the price ratio is constant, the MRTS is constant for cost-minimization input choices for any scaling factor such that $x \to \lambda \cdot x$.

2. For Leontief production function:

$y = \min(ax_1, bx_2)$



the optimal input ratio is given by:

$$\frac{x_2}{x_1} = \frac{a}{b}$$

For any price ratio, a change in input choice ratio $\frac{x_2}{x_1}$ does not depend on the price ratio. And at an optimum (cost minimization), we know from FOC that price ratio = MRTS. Thus the elasticity of substitution is:

$$\sigma_{12} = \frac{d(x_2/x_1)/(x_2/x_1)}{d(MRTS)/(MRTS)} = \frac{d\ln(x_2/x_1)}{d\ln(P_1/P_2)} = 0$$

3. Since more factors are mobile in the long run (LR), the total LR costs are smaller than total short run (SR) costs. There must be some optimum w for which $LRC(w) = SRC(w)$, and after this the difference increases with w. So, define

$h(w) = LRC(w) - SRC(w)$, h is concave in w.

Thus,
$$\frac{\partial^2 h}{\partial w_i^2} = \frac{\partial^2 LRC(w)}{\partial w_i^2} - \frac{\partial^2 SRC(w)}{\partial w_i^2} \leq 0$$

$$\therefore \quad \frac{\partial x_i^{LR}(w,y)}{\partial w_i} \leq \frac{\partial x_i^{SR}(w,y)}{\partial w_i}$$

4. Profit maximizing:

$$\max_{x_1, x_2} \pi = p\cdot y - w_1 x_1 - w_2 x_2 = p\cdot x_1^{\alpha}\cdot x_2^{\beta} - w_1 x_1 - w_2 x_2$$

FOCs: $$\frac{\partial \pi}{\partial x_1} = p\cdot \alpha \cdot x_1^{\alpha-1}\cdot x_2^{\beta} - w_1 = 0$$

$$\frac{\partial \pi}{\partial x_2} = p\cdot \beta\cdot x_1^{\alpha}\cdot x_2^{\beta-1} - w_2 = 0$$

Solve x_1 and x_2 from the above two equations:

$$x_1^* = \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \cdot \left(\frac{w_1}{\alpha}\right)^{\frac{1-\beta}{\alpha+\beta-1}} \cdot \left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}}$$

$$x_2^* = \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \cdot \left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}} \cdot \left(\frac{w_2}{\beta}\right)^{\frac{1-\alpha}{\alpha+\beta-1}}$$

Then we get the profit function:

$$\begin{aligned}\pi(p, w_1, w_2) &= p\cdot (x_1^*)^{\alpha}\cdot (x_2^*)^{\beta} - w_1\cdot x_1^* - w_2\cdot x_2^* \\ &= \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \cdot \left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}} \cdot \left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}} - \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \cdot w_1 \cdot \left(\frac{w_1}{\alpha}\right)^{\frac{1-\beta}{\alpha+\beta-1}} \cdot \left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta-1}} \\ &\quad - \left(\frac{1}{p}\right)^{\frac{1}{\alpha+\beta-1}} \cdot w_2 \cdot \left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta-1}} \cdot \left(\frac{w_2}{\beta}\right)^{\frac{1-\alpha}{\alpha+\beta-1}}\end{aligned}$$

For a well-defined production function, we must have $y' > 0$ and $y'' < 0$

so $\frac{\partial y}{\partial x_1} = \alpha \cdot x_1^{\alpha-1}\cdot x_2^{\beta} > 0$, $\quad \frac{\partial^2 y}{\partial x_1^2} = \alpha(\alpha-1)\cdot x_1^{\alpha-2}\cdot x_2^{\beta} < 0$

$\therefore$ $\alpha > 0$ and $\alpha < 1$

Similarly, it requires $\beta > 0$ and $\beta < 1$

5. The inputs in the following technology are perfect substitutes:

$$y = \sum_{i=1}^{n} a_i x_i$$

so cost minimization entails the input with the lowest input price will be used.

Thus $y = a_i x_i \Rightarrow x_i = \frac{y}{a_i}$

Thus $c(w, y) = \min\{w_i x_i\} = \min\{\frac{w_i}{a_i} y\}$

6. Superadditivity of cost function means:

$$c(w^1 + w^2, y) \geq c(w^1, y) + c(w^2, y)$$

By definition of the cost function, there must be some cost minimizing bundles for each price vector:

$$c(w^1+w^2, y) = \min_x \{(w^1+w^2)\cdot x^*\} \quad s.t.\ f(x)=y$$

$$c(w^1, y) = \min_x \{w^1 \cdot x'\} \quad s.t.\ f(x)=y$$

$$c(w^2, y) = \min \{w^2 \cdot x'\} \quad s.t.\ f(x)=y$$

From cost minimization:

$$c(w^1, y) \leq w^1 \cdot x^*$$

$$c(w^2, y) \leq w^2 \cdot x^*$$

$$c(w^1+w^2, y) = (w^1+w^2)\cdot x^* = w^1 \cdot x^* + w_2 \cdot x^* \geq c(w^1, y) + c(w^2, y)$$

Hence, $c(w^1+w^2, y) \geq c(w^1, y) + c(w^2, y)$

To show the cost function is non-decreasing in w:

Suppose $\Delta w \geq 0$, it could be $\forall i\ \Delta w_i \geq 0$ or $\exists i\ \Delta w_i \geq 0$

For $\Delta w_i = 0$, we have $c(\Delta w_i, y) = 0$

Then by superadditivity:

$$c(w+\Delta w, y) \geq c(w, y) + c(\Delta w, y)$$

$$\therefore c(w+\Delta w, y) \geq c(w, y)$$