Econ 304a

Blake LeBaron

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Lecture notes on Arrow/Debreu securities and stochastic discount factors

Arrow/Debreu (A/D) securities are assets that payoff in a specific state of nature, at a certain time. They are useful building blocks to think about other assets even though they don't really exist. (Although you can think of some things that have an A/D nature to them.)

Let $p_{t,s,z}$ be the price of an A/D security at time t, paying $1 at time s, in state z. (Assume there are Z states of the world, indexed by z.)

We will simplify this to $p_{s,z}$ for the rest of these notes to minimize on subscripts. Always assume that, t = today, the present period in terms of when the price is observed.

What is a state? Could be anything: earthquake, rain, stock crash,

In a complete markets world, a common security, like a stock can be viewed as a package of payoffs in different states of the world. Let $q^j_{t+1,z}$ be a the payoffs in state z, time $t+1$. Assume this security produces no further payoffs in the future.

Using A/D securities we could build a portfolio that exactly replicates this stock. It would have $q^j_{t+1,z}$ "shares" of the A/D security for each state z. The price of such a portfolio would be,

$$p^* = \sum_{z=1}^{Z} p_{t+1,z} q^j_{t+1,z}.$$

Arbitrage principles say that the price of any two things paying the same amount in every state should be the same. The price of the stock, p^j_t should equal, p^*.

$$p^j_t = \sum_{z=1}^{Z} p_{t+1,z} q^j_{t+1,z}$$

This is an example of how to price a security using the A/D building blocks. (Building block is the right way to think about them.)

Now, define the following,

$$m_{t+1,z} = \frac{p_{t+1,z}}{\pi_z}$$

π_z is the probability of state z occuring.

Substitute this in,

$$p^j_t = \sum_{z=1}^{Z} \pi_z m_{t+1,z} q^j_{t+1,z}$$

$$p^j_t = E(m_{t+1} q^j_{t+1})$$

Also,

$$1 = E(m_{t+1}(1 + r^j_{t+1}))$$

This says that complete markets and absence of arbitrage alone imply that there will be an m_{t+1} that can be used to help "price" all assets. It is important to remember, that m_{t+1} is the same for any security j. So if you gave me m_{t+1} I could figure out the price of any asset in world. This has nothing to do with preferences.

m_{t+1} isn't free to be anything. As a random variable it must be ≥ 0. This comes from its construction,

$$m_{t+1,z} = \frac{p_{t+1,z}}{\pi_z}$$

The price of an A/D security cannot be negative since it is a positive payment in some state of nature. (It can be zero if it is a state that has low probability, or you don't care about, but a negative price implies that someone is paying you to get payed a \$1 in a future state.)

If one further assumes that a reprentative agent exists, then m_{t+1} becomes the marginal rate of substitution, but the point here is that m is more general.

It is referred to as the **Stochastic Discount Factor**. It also turns out that all common asset pricing relations (CAPM, APT,) can be stated in this framework.

Risk neutral pricing

$$1 = \sum_{z=1}^{Z} \pi_z m_{t+1,z}(1 + r)$$

Let

$$\pi^*_z = \pi_z(1 + r)m_{t+1,z}$$

Call this a kind of pseudo probability. It adds up to one, and is greater than zero.

The pricing relationships we have worked about before can be written in terms of this as

$$p^j_t = \frac{1}{1 + r} \sum_{z=1}^{Z} \pi^*_z q^j_{t+1,z}$$

This is a risk neutral like pricing relationship in the pseudo probability measure. It says that you can create a kind of fake probability measure that will price securities as if the world were risk neutral. This is very common in option pricing.

Lecture Notes on Complete Market Aggregation

Assume the following things:

1. Complete Markets
2. CRRA preferences
3. Common preference parameters
4. All agents are at interior solutions (first order conditions are satisfied)

Each individual maximizes,

$$U_t^i = \sum_{s=t}^{\infty} \beta^s \frac{1}{1-\gamma} c_{s,i}^{1-\gamma}$$

Euler equation for person i, asset j,

$$c_{t,i}^{-\gamma} p_t^j = E_t(\beta c_{t+1,i}^{-\gamma} q_{t+1}^j)$$

This applies to the Arrow/Debreu security too,

$$c_{t,i}^{-\gamma} p_{t+1,z} = \pi_z \beta c_{t+1,i,z}^{-\gamma}$$

π_z is the probability of state z occuring.

Algebra gives,

$$c_{t,i} p_{t+1,z}^{-1/\gamma} = (\pi_z \beta)^{-1/\gamma} c_{t+1,i,z}$$

Sum over all people i, and divide by the population to get,

$$c_t p_{t+1,z}^{-1/\gamma} = (\pi_z \beta)^{-1/\gamma} c_{t+1,z}$$

where c_t is now per capita consumption.

Algebra gives,

$$c_t^{-\gamma} p_{t+1,z} = \pi_z \beta c_{t+1,z}^{-\gamma}$$

However, this is just the FOC for the A/D security, using a percapita consumption representative agent.

$$p_{t+1,z} = \pi_z \beta \frac{c_{t+1,z}^{-\gamma}}{c_t^{-\gamma}}$$

Now use the A/D security to “price” a security j paying $q_{t+1,z}^j$ in state z.

$$p_t^j = \sum_{z=1}^{Z} p_{t+1,z} q_{t+1,z}^j$$

$$p_t^j = \sum_{z=1}^{Z} \pi_z \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} q_{t+1,z}^j$$

$$p_t^j = E(\beta \frac{u'(c_{t+1,z})}{u'(c_t)} q_{t+1}^j)$$

$$p_t^j = E(m_{t+1} q_{t+1}^j)$$

or

$$1 = E(m_{t+1}(1 + r_{t+1}^j)$$

If π were probabilities conditional on time t information, this would be,

$$p_t^j = E_t(m_{t+1} q_{t+1}^j)$$
$$1 = E_t(m_{t+1}(1 + r_{t+1}^j)$$