Fin285a: Time Aggregation and Long Horizons
Section 6.1
Holding periods
Holding periods

- Assume portfolio stays constant
- Evaluate portfolio $h$ periods in future
- Examples
  - 10 day $VaR$
  - 1 month $VaR$
  - 1 year $VaR$
- Other complications: present values and discounting
Long range returns

- Long horizon wealth is approximately log normal
- Not symmetric
- Mean $\neq$ median
Wealth at long horizons

Wealth is approximately log normal

Why?

Example:

⇒ Start with 100 today
⇒ Assume returns are independent

\[ W_T = 100 \times \prod_{t=1}^{T} (1 + R_t) \]

\[ \log(W_T) = \log(100) + \sum_{t=1}^{T} \log(1 + R_t) = \log(100) + \sum_{t=1}^{T} r_t \]

By Central Limit Thm: \( \log(W_t) \sim \text{Normal} \)
Long horizon returns

\[(1 + R_t(T)) = \prod_{i=0}^{T-1} (1 + R_{t-i}) \quad (6.1.1)\]

\[\log(1 + R_t(T)) = \sum_{i=0}^{T-1} \log(1 + R_{t-i}) \quad (6.1.2)\]

\[\mu = E(\log(1 + R_t)), \quad \sigma^2 = \text{var}(\log(1 + R_t))\]
Long horizon returns

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(1 + R_t(T)) = \prod_{i=0}^{T-1} (1 + R_{t-i}) \tag{6.1.1}
\]

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Reminder, independent RV's: \(\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)\)
**Long horizon returns**

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(1 + R_t(T)) = \prod_{i=0}^{T-1} (1 + R_{t-i}) \quad (6.1.1)
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Reminder, independent RV's: \(\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)\)

\[
\log(1 + R_t(T)) \sim N(T\mu, T\sigma^2) \quad (6.1.3)
\]
What should investors care about?

- $E(W_T) = 100 \prod_{t=1}^{T} E(1 + R_t)$
- Since $W_T$ is right skewed, this is not the median of $W_T$
- See longhmean.py
Independence of long range returns

• How independent are returns over time?
  ⇒ Rough approximation
  ⇒ Related to predictability
  ⇒ Not too bad at long horizons

• Deviations
  ⇒ Volatility
  ⇒ Trends/momentum
  ⇒ Long range mean reversion (P/D, P/E ratios, real exchange rates)

• Reminder: Not cross sectional independence
VaR: Long sampling
Long sampling

- Sample data over appropriate horizon
- Estimate VaR using method of your choice
- Python: `histhd.py`
• Advantages
  ➨ Easy
  ➨ Exactly replicates long horizon distributions
  ➨ Replicates some time dependency
Long sampling summary

• Advantages
  ⇒ Easy
  ⇒ Exactly replicates long horizon distributions
  ⇒ Replicates some time dependency

• Disadvantages
  ⇒ Reduces sample size
  ⇒ Less precise estimates
Extended 1 period returns: Analytic
Quick reminder on independent RV’s

Note: IID = independent, identically distributed

\[ x_1 \sim N(\mu, \sigma^2) \]
\[ x_1 \sim N(\mu, \sigma^2) \]
\[ E(x_1 + x_2) = 2\mu \]
\[ \text{var}(x_1 + x_2) = 2\sigma^2 \]
\[ \text{std}(x_1 + x_2) = \sqrt{2\sigma} \]
Normal profits and losses

\[ P_{t+h} = P_t + \sum_{i=1}^{h} x_{t+i}, \quad x_{t+i} \sim N(\mu, \sigma^2) \]

\[ Q_{t+1} = P_{t+1} - P_t = x_{t+1} \]

\[ Q_{t+h} = P_{t+h} - P_t = \sum_{i=1}^{h} x_{t+i} \]

\[ Q_{t+h} \sim N(h\mu, h\sigma^2) \]

Find appropriate quantile of this distribution.
Continuously compounded (geometric) returns

\[ r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}) = \log(1 + R_t) \quad (6.1.4) \]

\[ r_t(h) = \log(1 + R_t(h)) = \log(1 + R_t) + \ldots + \log(1 + R_{t-h+1}) \quad (6.1.5) \]

\[ r_t(h) = r_t + r_{t-1} + \ldots + r_{t-h+1} \quad (6.1.6) \]

\[ E(\log(1 + R_t)) = E(r_t) = \mu, \quad \text{var}(\log(1 + R_t)) = \text{var}(r_t) = \sigma^2 \quad (6.1.7) \]

\[ E(r_t(h)) = h\mu, \quad \text{var}(r_t(h)) = h\sigma^2 \quad (6.1.8) \]

\[ \text{std}(r_t(h)) = \sqrt{h}\sigma \quad (6.1.9) \]
Continuously compounded (geometric) returns

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“Square root of time law”
Continuously compounded (geometric) returns

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“Square root of time law”

Independence, independence, independence!!!
Continuously compounded (geometric) returns

\[ r^* = q_p(r(h)) \]  \hspace{1cm} (6.1.10)

Use normality mean and std with \( h \), and \( \sqrt{h} \) adjustments.

\[ rstar = \text{stats.norm.ppf}(0.01, \text{loc} = h*\mu, \text{scale} = \text{np.sqrt}(h)*\sigma) \]
Continuously compounded (geometric) returns

\[ r^* = q_p(r(h)) \]  \hspace{1cm} (6.1.10)

Use normality mean and std with \( h \), and \( \sqrt{h} \) adjustments.

\[ r_{star} = \text{stats.norm.ppf}(0.01, \text{loc} = h \times \mu, \text{scale} = \text{np.sqrt}(h) \times \sigma) \]

\[ P^*(h) = P_t e^{r^*} \]  \hspace{1cm} (6.1.11)

\[ \text{VaR}_{h}(p) = P_t - P^*(h) = P_t - P_t e^{r^*} \]  \hspace{1cm} (6.1.12)
Extended 1 period returns: computational
Historical/bootstrap

• Method 1
  ⇒ Assume 1 day returns are independent over time
  ⇒ Aggregate to h period returns (draw random h periods)

• Method 2
  ⇒ Bootstrap 20 day blocked returns
  ⇒ Replicates dependence

• Python: histhdgeo.py
Expected Shortfall
Expected shortfall

- Compare $h$ day sample, independent bootstrap, blocked bootstrap
- Python: `histhdes.py`
Expected shortfall warning

• You can **NOT** use the expected shortfall formula when returns are log normal (recheck notes from estimating ES, last section)

• This means that you cannot:
  ➔ Use the $\sqrt{h}$ rule in a delta-normal situation
  ➔ With the special ES formula for normals
  ➔ You can only do this if you assume the $h$ period arithmetic returns are normal

• See code `histhdes.py`
The two bootstrap methods work fine, but you need to be consistent

When you look at log returns, build them back with

\[ P_{t+h} = P_t e^{\sum_{j=1}^{h} r_{t+j}} = P_t \prod_{j=1}^{h} e^{r_{t+h}} \]

And with arithmetic returns

\[ P_{t+h} = P_t \prod_{j=1}^{h} (1 + R_{t+j}) \]
Adjusting for Expected Returns
Adjusting for expected returns

- For short horizons, variance dominates expected returns
- U.S. stocks, annual, $\mu = 0.10$, $\sigma = 0.20$
- Daily $\mu \approx 0.10/250 = 0.0004$, $\sigma \approx 0.20/\sqrt{250} = 0.0126$
- At short horizons, means not too important for risk estimates
- Python: `VaRnomean.py`
Method summary
Method summary

• Assume continuous compounded returns are normal. Can use formulas.

• Direct long samples (h period returns, historical).

• Bootstrap:
  ➞ Independent bootstrap
  ➞ Bootstrap blocks/subsampling

• Other
  ➞ Fancier bootstraps
  ➞ Parametric models
Overview