Problem 2.42 The Entropy of a Black Hole

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A working definition of a **black hole** could be: "something from which nothing escapes". This definition will be sufficient for an analysis of the entropy of a black hole. To determine the entropy, later, we will need to know the radius of the black hole. This can be found through dimensional analysis.

Dimensional analysis is a powerful tool that can lead to accurate results quickly. First, we need to determine the parameters on which the radius of the black hole depends. These should be G, M because it is a gravitationally bounded system, and something else that will make the units come out correctly in m. It turns out that this other quantity could be the speed of light, c, which makes sense, because it acts as a maximum upper-bound. This leads to the conclusion that

$$r_{blackhole} \approx \frac{G \cdot M}{c^2}. \tag{1}$$

We can verify this equation by checking that the result is in meters. The units of each component are

$$\begin{aligned} G &= N \cdot m^2/kg^2 \\ M &= kg \\ c &= m/s \\ N &= \frac{kg \cdot m}{s^2}. \end{aligned} \tag{2}$$

Using these units, Eq.(1) becomes

$$r_{blackhole} \approx \frac{\frac{kg \cdot m}{s^2} \cdot m^2/kg^2 \cdot kg}{(m/s)^2} = m. \tag{3}$$

Eq.(1) can also be used to estimate the radius of a one-solar-mass black hole. Use the values

$$\begin{aligned} G &= 6.673 \times 10^{-11} N \cdot m^2/kg^2 \\ M &= 2 \times 10^{30} kg \\ c &= 2.998 \times 10^8 m/s. \end{aligned} \tag{4}$$

$$\tag{5}$$

Plugging these values into the formula gives

$$r_{blackhole} \approx \frac{6.673 \times 10^{-11} N \cdot m^2/kg^2 \cdot 2 \times 10^{30} kg}{2.998 \times 10^8 m/s^2} = 1483m \approx 1500m. \quad (6)$$

This result is a fairly large value, which contradicts the common belief that black holes are very small, very dense objects. It is correct to say that they are extremely dense, but we have just proven that they are not always small. The density would be enormous. The radius of the sun is $6.96 \times 10^8 m$, so its density would be $\approx 1400 kg/m^3$. The black hole would have a density of $\approx 1.5 \times 10^{20} kg/m^3$, or about 10^{17} times higher than the density of the sun.

Using the fact that particles do not escape from a black hole, we can say that the entropy of a black hole should be determined completely by the amount of mass that goes into making the black hole. We also know that all measurable properties of a system are either intensive or extensive. First, we know that energy is proportional to N, and therefore must be an extensive property. Using

$$du = Tds = Pdv, \quad (7)$$

we can extrapolate that since du is extensive, ds and S must be extensive as well. This is because T is intensive, and also because an extensive property times and intensive property gives an extensive property. This can be verified through examples involving basic thermodynamic properties. Because S is an extensive property, we know that it must be proportional to the mass of the system, which also means that it would be directly proportional to the number of particles, N.

For either a monatomic idea gas, or a high-temperature Einstein solid, we know that the entropy is given by k times a logarithm of the multiplicity (Schroeder, 79). Again, by similar examples, this shows that S must be proportional to N.

Now, to gain further insight on the system, we can show that the entropy must go to a maximum, and in order to do this, the system will act as if it was created from the maximum number of particles for a given amount of mass. The entropy will go to a maximum based on the second law of thermodynamics. After a black hole is formed, we cannot determine the type of mass that went into making the black hole. Since the entropy goes to a maximum, we can say that the entropy of the black hole must be greater than the entropy of any conceivable type of matter that could have been used to create the black hole (Schroeder, 83). This is because the back hole will create as many particles as it takes to maximize the entropy. Therefore, in order to determine the order of the entropy of a fixed-mass black hole, we must think about all possible ways it could have been created and choose the possibility with the greatest entropy. Suppose the black hole has a mass of $B\,kg$. Let N, be the number of particles, with mass b, that went into creating the black hole. Then the number of particles would be determined by

$$N = \frac{B}{b}. \tag{8}$$

To maximize the entropy, we need to maximize N because entropy is proportional to the number of particles. Therefore, b would have to be as small as possible.

In fact, black holes of maximum entropy are created from massless photons. This maximizes N. Specifically, photons of the lowest possible energy should be used (Schroeder, 84). The energy of the photons is given by

$$E_{photon} = \frac{h \cdot c}{\lambda}. \tag{9}$$

There will be N of these photons that go into making the black hole, and their total energy will be $M\,c^2$. Equating these two energies gives

$$M\,c^2 = \frac{h \cdot c}{\lambda} N. \tag{10}$$

λ can be no larger than the diameter of the black hole. Using Eq.(1), this means

$$\lambda_{max} = 2\frac{G \cdot M}{c^2}. \tag{11}$$

Insert the value for λ into Eq.(10) and solve for N. This gives

$$N = \frac{2G \cdot M^2}{h \cdot c}. \tag{12}$$

So, the entropy of a black hole is given by

$$S_{blackhole} = k\frac{2G \cdot M^2}{h \cdot c}. \tag{13}$$

A more rigerous calculation will give a factor of 8π rather than just the 2.

Eq.(13) can be used to calculate the entropy of a one-solar-mass black hole. This gives an entropy of $S = 1.5 \times 10^{54} J/k$. We have seen that The entropy, S, is given by

$$S \sim N \cdot k. \tag{14}$$

Therefore, we see that the number of particles would be on the order of 10^{79}. Although this is a big number, it is not of the same order as a very big number, which would be more like $10^{10^{79}}$.

For a comparison, the entropy of the sun is around 10^{33}. This means that the entropy of about 10^{20} times that of our sun is contained in our model black hole. Since there are only about 10^{11} stars in our Milky Way, this means that it would take about 10^{9} galaxies to create the same entropy as that in the black hole. This amount of entropy is packed into a sphere of radius of only $1.5\,km$!

Thus, some insight on the creation of a black hole and its entropy has been gained. Since objects can never escape from a black hole, its entropy will never decrease. This makes the process of an item entering a black hole an irreversible process.