Problem 1.16 The Exponential Atmosphere

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Using a few differential equations derived from the ideal gas law, $PV = NkT$, it is possible to derive a function describing the variation of pressure with altitude in terms of the density of air. In a similar manner, the density of air as a function of altitude can be derived. As an intermediate step, the barometric equation will be derived.

First, we will consider a horizontal slab of air (at rest) has thickness dz and area A. The forces on the slab are the forces due to the pressure from above and below, and its weight.

The vertical summation of forces for a body at rest is given by

$$\sum F_y = 0. \tag{1}$$

Using this concept, we can write

$$AP_{above} + W = AP_{below}. \tag{2}$$

Let

$$W = A\rho dz. \tag{3}$$

Using Eq.(2) with Eq.(3) we have

$$P_{above} + g\rho dz = P_{below}. \tag{4}$$

Letting $P_{above} - P_{below} = dP$, it can be shown that

$$\frac{dP}{dz} = -g.\rho \tag{5}$$

Now to write ρ in terms of P, T, and the average mass of the air molecules (m), we will need the ideal gas law, which is given by

$$PV = NkT. \tag{6}$$

Density is the mass of all molecules in the slab divided by the volume of the slab, or

$$\rho = \frac{Nm}{V}. \tag{7}$$

Using Eq.(6), we obtain

$$\rho = \frac{Pm}{kT}. \tag{8}$$

Inserting this result into Eq.(5) we obtain the **barometric equation** given by

$$\frac{dP}{dz} = -\frac{mg}{kT}P. \tag{9}$$

Assuming the temperature of the atmosphere is independent of height, we can solve the barometric equation to obtain the pressure as a function of height. Solving the differential equation given in Eq.(9), using a separation of variables technique, we have

$$\frac{dP}{P} = -\frac{mg}{kT}zdz, \tag{10}$$

or

$$\int_0^P \frac{dP}{P} = \int_0^z -\frac{mg}{kT}zdz, \tag{11}$$

or

$$e^{ln(P)} = e^{\frac{-mg}{kT}z} + C. \tag{12}$$

Now let $C = P(0)$ because when $z = 0$, $P = C$. So,

$$P(z) = P(0)e^{\frac{-mg}{kT}z}. \tag{13}$$

Similar to Eq.(13), we can write an expression for the density of air in the atmosphere as a function of the height z. Rearranging Eq.(6) gives

$$\rho = \frac{N}{V} = \frac{P}{kT}. \tag{14}$$

Replacing P with ρ and cancelling the $\frac{1}{kT}$ gives

$$\rho(z) = \rho(0)e^{\frac{-mg}{kT}z}. \tag{15}$$

Finally, the result Eq.(13) can be used to estimate the pressure in atmospheres at various locations around the world. Assume the pressure at sea level $P(0)$ is 1 atm. Also, assume the temperature is room temperature (298K). Take the average mass to be the average mass of a mole of dry air. This is $29u = 4.82 \times 10^{-26}Kg$.

The following table shows the pressure for various locations.

Thus we have observed that both pressure and density are exponential functions of the altitude. To illustrate this concept, notice that even though the altitude of Mt. Everest is twice that of Mt. Whitney, the pressure is not simply half. Rather, it is related by an exponential form.

Location	$z(m)$	$P(z) atm$
Ogden, Utah	1430	.85
Leadville, Colorado	3090	.70
Mt. Whitney, California	4420	.60
Mt. Everest, Nepal	8850	.36