Problem 1.39 The Speed of Sound in a Medium

Bobby Rohrkemper
April 14, 2003

Using the bulk modulus of a medium and the ideal gas law, it is possible to compute the speed of sounds in an ideal gas. The bulk modulus is a measure of the stiffness of a medium. This analysis raises several interesting questions about isothermal and adiabatic processes and also yields an explanation for why Scotland's Battlefield Band played out of tune in Utah.

First, we need to know the definition of **bulk modulus (B)**.

$$B \equiv \frac{\Delta P}{\frac{-\Delta V}{V}}. \tag{1}$$

B can also be written in differential form. This gives

$$B = \frac{dP}{\frac{-dV}{V}}. \tag{2}$$

B for an ideal gas can be determined for both isothermal and adiabatic compressions. Isothermal compressions occur when the temperature (T) is constant. Start with the ideal gas law given by

$$PV = NkT. \tag{3}$$

The temperature constraint can be expressed in terms of partial derivatives. Finding $(\frac{\partial P}{\partial V})_T$ and inserting the expression into Eq.(2) will give B in terms of pressure P. So, using Eq.(3) yields

$$P = \frac{NkT}{V}, \tag{4}$$

or

$$dP = -\frac{NkT}{V^2}dV, \tag{5}$$

or

$$\frac{dP}{dV} = \frac{-NkT}{V^2}, \tag{6}$$

and replacing $\frac{dP}{dV}$ in Eq.(2) with Eq.(6) gives

$$B = -\frac{-NkT}{V^2}, \tag{7}$$

or

$$B = \frac{NkT}{V} = P. \tag{8}$$

Similarly, an adiabatic process can by analyzed by using the definition of an adiabatic process. It is one in which no heat is transferred. Therefore, $Q = 0$. So,

$$\Delta U = Q + W = W. \tag{9}$$

Starting with the equipartition theorem, it can be shown (Schroeder, 26) that

$$V^\gamma P = C, \tag{10}$$

where

$$\gamma = (f + 2)/f. \tag{11}$$

Next, rearrange to give

$$P = \frac{C}{V^\gamma}. \tag{12}$$

Differentiating Eq.(12) yields

$$dP = \frac{-C\gamma}{V^{\gamma+1}} dV. \tag{13}$$

Since $-dP/dV = B$,

$$B = -V \frac{-C\gamma}{V^{\gamma+1}}, \tag{14}$$

or

$$\frac{C\gamma}{C/P} = \gamma P \tag{15}$$

Now, use Eq.(10) to plug in for V^γ. This gives

$$\frac{C\gamma}{C/P}. \tag{16}$$

Therefore,

$$B = \gamma P \tag{17}$$

gives the value of B for an adiabatic gas.

It will later be shown that the speed of sound can be computed from B. First, a decision must be made to use the isothermal or adiabatic value for B. As a sound wave traverses a medium, it causes work to be done on the system through expansion and compression of the system. This can be directly measured through a change in pressure. In fact, a microphone can be used to measure a change in pressure. Since a sound wave travels very quickly ($V = 340m/s$), the process happens very quickly, and no heat is exchanged. There is a change in temperature due to the expansion and contraction, but the temperature change is not maintained. Because no heat is transferred, the adiabatic method should be used. Because there is a temporary change in T, an Isothermal model should not be used.

Applying Newton's laws to the oscillations of a continuous medium will yield the speed of sound C_s

$$C_s = \sqrt{\frac{B}{\rho}}. \tag{18}$$

Note that ρ is the density of the medium. So,

$$C_s^2 = \frac{B}{\rho}, \tag{19}$$

or

$$\rho C_s^2 = B = \gamma P. \tag{20}$$

Using the ideal gas law to plug in for P, and m/V for ρ gives

$$\frac{m}{V} C_s^2 = \frac{\gamma(NkT)}{V}. \tag{21}$$

$f = 5$ for air (Schroeder, 25), so we can write

$$mC_s^2 = \frac{(5+2)}{5} kT, \tag{22}$$

where γ is given by Eq.(11). Simplifying gives

$$C_s = \sqrt{\frac{7}{5m} kT}. \tag{23}$$

The speed of sound can now be found numerically using $m = 28u = 4.65 \times 10^{-26} Kg$, $k = 1.381 \times 10^{23} J/K.$, and $T = 300K$. This gives $C_s = 246m/s$. The ν_{rms} formula (Schroeder, 13) gives a slightly higher result of

$$\nu_{rms} = \sqrt{\frac{3kT}{m}} = 353m/s. \tag{24}$$

The ν_{rms} is most likely higher because it takes into account the various directions the molecules are moving in. The formula we derived in (23) only takes into account the direction of the propagation of sound. It should also be noted that the velocity of a single air molecule is much larger than the speed of sound. The value of ν_{rms} is lower than the speed of a single air molecule because the molecules move in many directions.

This equation shows that C_s has a temperature dependence. This temperature dependence has an application in acoustics. Suppose a bagpiper plays in tune at sealevel and then notices that his instrument has gone out of tune at an altitude. The explanation of why this happens can be complicated because there are many factors to account for. First note that in Eq.(23) a temperature decrease causes a decrease in the speed of sound. Therefore, at a higher altitude, where the temperature is lower, C_s will also be lower. Supporting this observation is the Ideal Gas Law given by Eq.(3) where a decrease in T corresponds with a decrease in P at a roughly constant V. We expect a decrease in P because it has been proven and commonly observed that P decreases with altitude. To draw a connection to the bagpipe, note that

$$V_{sound} = \lambda\nu, \tag{25}$$

where ν is the frequency, and λ is the wavelength. For a bagpipe, in order for the note to resonate, λ will be fixed since the diameter and length of the pipes are fixed. Therefore, by Eq.(25), V_{sound} decreasing will cause the frequency to decrease. The note will go flat.

Thus, it has been shown that an adiabatic model of air can be used to compute V_{sound} and its variance with temperature. In fact, for any specific temperature, C_s can by computed with Eq.(23).