A White Dwarf

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A white dwarf star can be modelled as a gas of electrons, each with a corresponding proton and neutron. Modelling the star as a uniform-density sphere will allow us to determine a relationship between the mass and radius, and will also allow for calculations of the star's other attributes including its kinetic energy. Assuming $T = 0$ will an appropriate approximation. This is necessary in order to keep all of the particles in the spherical momentum space.

First, we should derive the basic form for the gravitational potential energy. We can do this through dimensional analysis. We know that U_grav must depend on the mass, M, the gravitational constant, G, and the radius, R. The correct units are given when

$$U_{grav} = -(constant)\frac{GM^2}{R}. \tag{1}$$

This formula can also be derived in more detail through an integration of the gravitational force. First, find the work needed to bring a thin spherical shell from infinity into a distance, r—the radius of the sphere built so far. Then, find the work needed to move the shell out to its final location at radius R. The work to do this is given by

$$W = 3/5\,\frac{GM^2}{R} \tag{2}$$

We know that work is the negative of potential energy, so the value of the constant must be 3/5.

To calculate the star's kinetic energy, we must decide if the electrons are non-relativistic or relativistic. We will try both cases and determine the best answer through a numerical analysis.

First, try the non-relativistic case. The density of states in this case can be derived by using $\epsilon = p/2m$ and is given by

$$g(\epsilon)d\epsilon = \frac{2\epsilon^{1/2}Vm^{3/2}\pi 8}{h^3\sqrt{2}}. \tag{3}$$

We can integrate to find the number of electrons that lie between energy zero and the fermi energy.

$$N_e = \int_0^{\epsilon_f} g(\epsilon)d\epsilon. \tag{4}$$

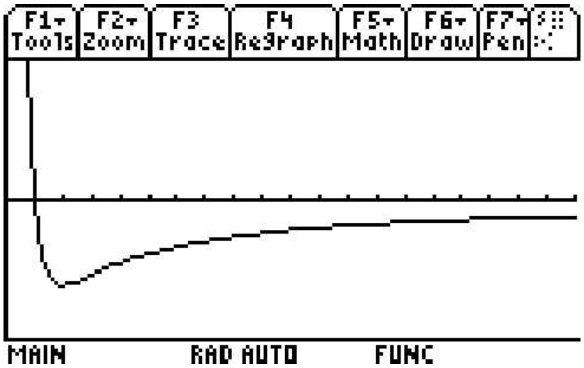

Figure 1: The total energy as a function of the star's radius

Then, solve this equation for ϵ_f and insert this value into the result of the following.

$$U_{kinetic} = \int_0^{\epsilon_f} \epsilon g(\epsilon) d\epsilon. \tag{5}$$

Next, plug in $V = 4/3\pi R^3$ and the number of electrons, $N_e = \frac{M}{2m_p}$. The result is

$$U_{kinetic} = (0.0088)\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2} = \frac{3}{5}\epsilon_F N. \tag{6}$$

Equipped with formulas for both U_{grav} and $U_{kinetic}$, we can find the equilibrium radius by minimizing the total energy. The general form will be

$$E_{total} = -1/R + (constant)/R^2. \tag{7}$$

To solve for the equilibrium radius, we take the derivative of E_{total} and set it equal to zero.

This gives the equilibrium radius as a function of the star's mass—

$$R = \frac{0.03}{G}\frac{h^2}{m_e m_p^{5/3} M^{1/3}}. \tag{8}$$

This means that as the radius decreases, the mass will increase. This makes intuitive sense because the potential energy will decrease more than the kinetic energy decreases as mass is added.

Inserting values for the constants yields $R = 7.3 \times 10^6\ m$. A sphere of this radius and a mass of one-solar mass will have a density of $\rho = 1.3 \times 10^9\ kg/m$. This is roughly a million times the density of water.

Once integrated, Eq.(4) can be solved for the fermi energy. This gives

$$\epsilon_F = \frac{h^2}{8m_4}{\frac{3N_e}{\pi V}}^{2/3}. \tag{9}$$

Inserting for the constants gives $\epsilon_F = 1.9 \times 10^5 eV$.

The fermi temperature is then given by the condition

$$T_F = \frac{\epsilon_F}{k} = 2.2 \times 10^9 K. \tag{10}$$

This model works well for temperatures below this value. Therefore, we can say that the approximation $T = 0$ is valid since it is not likely that the temperature will be above ϵ_F.

Suppose instead that the behavior of the electrons is relativistic. Now, to derive the density of states, we will need to use $\epsilon = pc$. In a previous derivation, we found

$$g(\epsilon)d\epsilon = \frac{8\pi\epsilon^2 V}{h^3c^3}. \tag{11}$$

This time, solving Eq.(4) for ϵ_F gives

$$\epsilon_F = \frac{3}{8}{\frac{h^3c^3N}{\pi V}}^{1/3}. \tag{12}$$

Then, a similar procedure to the non-relativistic case tells us that

$$U_{kinetic} = \frac{4}{3}N\frac{3}{8}{\frac{h^3c^3N}{\pi V}}^{1/3}. \tag{13}$$

The total kinetic energy is then proportional to $1/R$ because there is a R^3 inside the $V^{1/3}$. We also know that the gravitational potential energy is proportional to $1/R$. The form will then be

$$E_{total} = -1/R + c/R. \tag{14}$$

Because this function does not have a minimum value, there will be no equilibrium radius for the case with ultrarelativistic electrons.

To be confident in our results, we should analyze the transition from the nonrelativistic regime to the ultrarelativistic. This occurs close to the rest energy of an electron. Note that we have shown that the nonrelativistic case is stable whereas the relativistic case is not. From Eq.(6), we know

$$\bar{U} = \frac{3}{5}U_{kinetic} = 10^5\ eV. \tag{15}$$

The rest energy of the same electron is roughly $5 \times 10^5\ eV$. The average kinetic energy is well below the rest energy of the electron. From this, we can conclude that a one-solar mass white dwarf star is stable in the nonrelativistic regime. To find the mass of the white dwarf when it first becomes relativistic, we can equate the fermi energy and the rest energy of the electron. Then, solve for the mass at this energy. This results in a mass of about 1.9 solar-masses. Above this mass, the white dwarf star would be relativistic and therefore nonstable.

In conclusion, we have worked through the calculation of the gravitational potential and the kinetic energy for both the relativistic and nonrelativistic schemes. We showed that the second is stable, but the first is not through a minimization of the total energy. Lastly, we analyzed the transition from one scheme to the next based on the mass of the star.