A POSTNIKOV APPROACH TO MANIFOLDS

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Abstract. We show that up to a Wall obstruction to surgering a cobordism to a simple h-cobordism, the diffeomorphism type of an m-Manifold is determined by the homotopy $[m/2]$-type of its normal bundle and a normal invariant.

1. Introduction

Here we investigate how much of the Kervaire, Milnor, Browder, Novikov, Wall, Ranicki surgery theory can be carried out

Let M be a smooth, closed compact m-manifold and ν the normal bundle of M embedded in R^{m+k}, k large. universal bundle and let

$$\begin{tikzcd}
\nu \arrow[r] \arrow[d] & \nu_n \arrow[r] \arrow[d] & \xi \arrow[d] \\
M \arrow[r] & M_n \arrow[r] & BO_k
\end{tikzcd}$$

be bundle maps where M_n is the nth stage in a Moore-Postnikov decomposition of $M \longrightarrow BO_k$. The map $\nu \longrightarrow \nu_n$ gives an element $\alpha_n \in \pi_{m+k}(T(\nu_n))$ (T=Thom space). We have two independent results. The simplest is the observation that surgery theory works the same on $(M_{[m/2]}, \nu_{[m/2]}, \alpha_{[m/2]})$ as on (M, ν, α). The other is a construction of $(M_{n+1}, \nu_{n+1}, \alpha_{n+1})$ from (M_n, ν_n, α_n) when $2n + 1 \geq m$.

By way of an introduction we carry out our program in considerable detail assuming M simply connected. All our spaces will be assumed to be homotopy equivalent to CW complexes of finite type. Baring some small details which we provide, reference to the books by Wall, *Surgery on Compact Manifolds* [5] and Raniski[4] suffice to generalize our program to non-simply connected manifolds, manifolds with boundary and PL manifolds.

In applying standard surgery theory, $\nu|M^{(t)}$ is unfortunate because there is no map from ν to $\nu|M^{(t)}$. We remedy this as follows: If $F : Y \longrightarrow Z$ is a map of spaces we denote the mapping cylinder by Z_f and often we denote the pair (Z_f, X) by (Z, X). Recall, a (Moore-Postnikov) n-factorization of f consists of maps $h : Y \longrightarrow X$ and $g : X \longrightarrow Z$ such that g is a fibre map, $f = gh$, $\pi_i(X, Y) = 0$ for $i \leq n+1$ and $\pi_i(Z, X) = 0$ for $i > n+1$. It is easy to construct such a factorization by attaching cells to X and then converting the resulting map, from the constructed space to Y, into a fibre map. We spell this out in section 3.

We define the n-type of a vector bundle by n-factoring its classifying map into BO_k. Define a k-plane bundle ζ , over X, to be a *vector bundle n-type* if, with respect to its classifying map, $g : X \longrightarrow BO_k$, $\pi_i(BO_k, X) = 0$ for $i > n+1$. We define an *n-type for a bundle* ξ over Z to be a bundle map $F : \xi \longrightarrow \zeta$ such that ζ is an n-type and $\pi_i(X_{F_Z}, Z) = 0$ for $i \leq n+1$. We say two k-plane bundles, ξ_1 and ξ_2 have the same n-type if they have n types $F_i : \xi_i \longrightarrow \zeta$ for some n-type ζ. One easily shows that having the same n-type is an equivalence relation and if ξ is a vector bundle over Z, a CW complex, ξ and $\xi|Z^{n+1}$ have the same n-type.

We define two closed, compact smooth m-manifolds M_1 and M_2 to have the same Thom homotopy n-type (in contrast to homotopy n-type) if there is a cobordism N between them, embedded in R^{m+k+1}, and a map F of its normal bundle ν_N into an n-type k-plane bundle ζ such that for $i = 1$ and 2, $F|(\nu_N|M_i)$ is an n-type for $\nu_N|M_i$. Our strong form of Ponicare duality for closed manifolds is then the following theorem:

Theorem1.1. *Suppose M_1 and M_2 are closed, compact smooth m-manifolds and $2n+1 \geq m$. If M_1 and M_2 have the same Thom homotopy n-type, then they have the same Thom homotopy k-type for all $k > n$.*

Thus if M_1 and M_2 are as Theorem 1 and of the same Thom homotopy n-type and $m > 4$, the obstruction to their being diffiomorphic is the Wall obstruction to a cobordism being a simple h-cobordism. By the main theorem in Browder's book, Surgery on Simply Conected Manifolds ([1]):

Corollary 1.2. *Suppose M_1 and M_2 are simply connected closed, compact smooth m-manifolds, $m > 4$ and $2n+1 \geq m$. Then, for m even, M_1 is diffeomorphic to M_2 (for m odd, M_1 is diffeomorphic to M_2 plus a homotopy sphere bounding a π manifold) if and only if M_1 and M_2 have the same Thom homotopy n-type.*

For a vector bundle ζ, let $\Omega_m(\zeta)$ be the cobordism group in the usual way, that is, cobordism classes of pairs $(M, F)$where M is a closed, compact smooth m-manifold and $F : \nu \longrightarrow \zeta$. For k large the Thom isomrphism theorem, $\Omega_m(\zeta) \approx \pi_{k+m}(T(\zeta))$ holds.

Throughout we use singular homology and cohomology and $H_q(X)$ denotes the singular homology of X with integer coefficients.

Definition 1.3. *Suppose X is a simply connected space. We define an element $\omega \in H_m(X)$ to be a Poincaré n-orientation if*

$$\cap\omega : H^{m-q}(X) \longrightarrow H_q(X)$$

is an isomorphism for $m - n \leq q \leq n$, an injection for $q = m - n - 1$ and an epimorphism for $q = n + 1$.

Remark The definition of a Poincaré n-orientation is easily extended to the analog of a manifold with boudary and to the non-simply connected case by introducing local coefficients.(see sections 4 and 5)

Lemma 1.4. *Suppose Z is a simply connected Poincaré m-complex with orientation $\beta \in H_m(Z)$ and $f : Z \longrightarrow X$ satisfies $\pi_i(X, Z) = 0$ for $i \leq n + 1$, $2n + 1 \geq m$. Then $f_*(\beta)$ is a Poincaré n-orientation for X.*

Proof. The lemma follows from $H_q(X, Z) = \pi_q(X, Z) = 0$ for $q \leq n + 1$ and the usual diagram:

$$\begin{tikzcd}
H^{m-q}(Z) \arrow[d] & H^{m-q}(X) \arrow[l] \arrow[d] \\
H^q(Z) \arrow[r] & H_q(X)
\end{tikzcd}$$

QED

Let $P_{m,n}(\zeta)$, ζ a k-plane bundle over X, be the set of elements $\{(M, F)\} \in \Omega_m(\zeta)$, $2n + 1 \geq m$, such that if [M] is an orientation of M, $F^*_M([M])$ is a Poincaré n-orientation of X. Note that when $n = m$, (X, ζ, α), $\alpha \in P_{m,n}(\zeta)$, is the starting point for Browder's and Wall's books on surgery theory ([1],[5]). Furthermore, the definition of Wall-Raniski's obsruction to α having a representative (M, F) such that F_M is a homotopy equivalence, $W : P_{m,m}(\zeta) \longrightarrow L_m(\pi_1(X))$, carries through without modification to $W : P_{m,n}(\zeta) \longrightarrow L_m(\pi_1(X))$, for $2n + 1 \geq m$, and same arguments yeild the following lemma which in turn gives a result similar to theorem 1.1.

Theorem 1.5. *If $m > 4$, and $\alpha \in P_{m,n}(\zeta)$, then α has a representative (M, F) such that F is an n-type for ν_M if and only if $2n+1 < m$ or $2n+1 \geq m$ and $W(\alpha) = 0$.*

We next turn to a result about the k-invariants of Poincaré complexes. We begin with some algebra. Suppose A and B are free chain complexes and $f : A \longrightarrow B$ is chain map. Let $c(f)$ denote the algebraic mapping cone of f, that is,

$$c(f)_q = B_q + A_{q-1}$$

$$\partial(u, v) = (\partial u + f(v), -\partial v).$$

Suppose $H_n(c(f)) = 0$. Let $\pi(f) = H_{n+1}(c(f))$ and $k(f)^{n+1} \in H^{n+1}(A; \pi(f))$ be the image of the idenity map under $\Theta^{-1} i^*$ where Θ is the usual map,

$$\Theta : H^{n+1}(c(f); \pi(f)) \approx Hom(H_{n+1}(c(f)), \pi(f))$$

and

$$H^{n+1}(c(f); \pi(f)) \xrightarrow{i^*} H^{n+1}(B; \pi(f))$$

where i is the inculsion of B into $c(f)$. These constructions are natural with respect to commutative diagrams,

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "h"] & B \arrow[d, "k"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

assuming $H_n(c(f)) = H_n(c(g)) = 0$. We denote the associated map by $(k, h) : c(g) \longrightarrow c(f)$. Also this diagram and the exact sequences of the form,

$$\longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(c(f)) \longrightarrow H_{n-1}(A) \longrightarrow$$

give the usual commutative ladder of exact seqences.

We define the cap product of a p-cochain V times a q-singular simplex T by

$$V \cap T = (-1)^{[(p-1)/2]} V(T_1) T_2$$

where T_1 is the first p-face of T and T_2 is the last $(q-p)$-face of T. Then,

$$\partial(V \cap c) = \delta V \cap c + (-1)^p V \cap \partial c$$

Suppose X is simply connected, $x \in \alpha \in H_m(X)$ and $2(n-1)+1 \geq m \geq 4$. Let $\bar{C}_*(X)$ be the chain complex with $\bar{C}_q(X) = C^{m-q}(X)$ and differential δ and let $P : \bar{C}_q(X) \longrightarrow C_q(X)$ be the chain map given by $P(V) = V \cap x$. Then we have an exact sequence,

$$\longrightarrow H_{q+1}(X) \longrightarrow H_{q+1}(c(P)) \longrightarrow H^{m-q}(X) \longrightarrow H_q(M) \longrightarrow$$

which yields:

Lemma 1.6. *$x \in H_m(X)$ is a Poincaré n-orientation if and only if $H_q(c(P)) = 0$ for $m-n \leq q \leq n+1$.*

Let $x \in H_m(X)$ be a Poincaré n-orientation, $D(X,x) = c(P)$, $\pi(x) = \pi(P) = H_{n+1}(c(f))$ and let $p : X(x) \longrightarrow X$ be the fibration with fibre the Eilenberg MacLane space $K(\pi(x), n)$ and k-invariant $k(x)^{n+1} = k(P)^{n+1} \in H^{n+1}(X; \pi(x))$.

Lemma 1.7. *Suppose Z is simply connected Poincaré m-complex with orientation α, $f : Z \longrightarrow X$ satisfiing $\pi_i(X,Z) = 0$ for $i \leq n$, $2(n-1)+1 \geq m$, and $x \in f_*(\alpha)$. Then x is a Poincaré n-orientation for X, f lifts to a map $f' : Z \longrightarrow X(x)$ satisfying $\pi_i(X(x), Z) = 0$ for $i \leq n+1$. For any other such lifting f'', f'' is homotopic to Gf', where $G : X(x) \longrightarrow X(x)$ is an homotopy equivalence over the identidy map on X.*

Remark If $x \in H_m(X)$ is a Poincaré n-orientation and $\pi_i(X) = 0$ for $i > n$, then X is potentially the n^{th} stage of a Postnikov decomposition of a Poincaré m-complex. If $m = n$, after killing off the homology above dimension m, X is a Poincaré m-complex. If $\hat{x} \in H_m(X(x))$ is such that $p_*(\hat{x}) = x$, then $\hat{x}$ is a Poincaré $n+1$ orentation if and only if $H_{n+1}(D(X(x), \hat{x}) = 0$. Alternatively, one can consider the Spivak nomal bundle of a Poincaré m-complex X and an n-factorization of its classifying map, $X \longrightarrow BG_k$, and carry out an analysis similar to the above for $M \longrightarrow BO_k$.

proof of 1.7. Let $\bar{f} : Z \longrightarrow \bar{X}$ and $p : \bar{X} \longrightarrow X$ be an n-factorization of f. Let $F : C_*(Z) \longrightarrow C_*(X)$ be the chain map induced by f. By standard homotopy theory (see section 3 for details), p is a fibration with fibre $K(\pi(F), n)$ and k- invariant $k(F)^{n+1} \in H^{n+1}(X; \pi(F))$ which is defined since $H_n(c(F)) \approx H_n(X,M) \approx \pi_n(X,Z) = 0$. Without loss of generality we may assume $x = f_*(y)$ where $y \in \alpha$. Consider the commutative diagrams of chain complexs,

$$\begin{tikzcd}
\bar{C}_*(X) \arrow[r, "P"'] \arrow[d, "h"] & C_*(X) \arrow[d] \\
C_*(Z) \arrow[r, "F"'] & C_*(X)
\end{tikzcd}$$

where $h(V) = (f)^*(V) \cap x$. This gives a commutative ladder:

$$\begin{tikzcd}
H^{m-n-1}(X) \arrow[r] \arrow[d] & H_{n+1}(X) \arrow[r] \arrow[d] & H_{n+1}(c(P)) \arrow[r] \arrow[d] & H^{m-n}(X) \arrow[r] \arrow[d] & H_n(X) \arrow[d] \\
H_{n+1}(Z) \arrow[r] & H_{n+1}(X) \arrow[r] & H_{n+1}(c(F) \arrow[r] & H_n(Z) \arrow[r] & H_n(X)
\end{tikzcd}$$

where the vertical arrows are isomorphisms since $H^{m-q}(X) \longrightarrow H_q(Z)$ is the complosition $H^{m-q}(X) \xrightarrow{f^*} H^{m-q}(Z) \xrightarrow{\cap\alpha} H_q(Z)$ which is an isomorphism for $m - q \leq n - 1$, that is, $q \geq n$. Hence $H_{n+1}(c(P)) \approx H_{n+1}(c(F))$ Then, by the naturality of $H^q(U; G) \to hom(H_q(X), G)$, the k-invariants of $X(x) \to X$ and of $\bar{X} \to X$ are isomorphic under (id, h). Hence $X(x) \longrightarrow X$ and $p : \bar{X} \longrightarrow X$ are fibre homotopically equivalent fibration. This proves the first part of Theorem 1.7.

Suppose f' and f'' are liftings of f. The fibration $X(x) \longrightarrow X$ is a principle bundle with fibre $K(\pi(x), n)$. Hence f''can be obtained from f' by right multipication by a map $t : Z \longrightarrow K(\pi(x), n)$. But since $H_q(X, Z) = 0$ for $q = n$ and $n - 1$, up to homotopy t pulls back to $s : X \longrightarrow K(\pi(x), n)$. Then right multipication of $id : X(x) \longrightarrow X(x)$ by sp gives the desired map G.

QED

Theorem 1.1 then follows from:

Corollary 1.8. *Suppose M is a simply connected smooth closed, compact m-manifold, $2(n-1)+1 \geq m$ and $F : \nu_M \longrightarrow \zeta$ is an $(n-1)$-type for ν_M and $[M] \in H_m(M)$ is the orientation of M. If $x \in (F_M)_*([M])$, then F_M lifts to a map $F'_M : M \longrightarrow X(x)$ giving a map $F' : \nu_M \longrightarrow p^*\zeta$. The map F' is an n-type for ν_M and for any other such lifting F'', F'' is homotopic to GF', where $G : p^*\zeta \longrightarrow p^*\zeta$ is a bundle map of $p^*\zeta$ to itself over a homotopy equivalence of X(x) to itself whixh in turn is over the idenity map on X. If (M_1, F_1) and (M_2, F_2) are cobordant, there is a cobordism between them which lifts to $p^*\zeta$*

Proof. The first part of 1.8 follows from 1.7 applied to $F_M : M \longrightarrow X$. Suppose N is a cobordism as in 1.8. By surgery below the middle dimension, one may make N such that $H_q(X, N) = 0$ for $q \leq [dimN/2]$. Hence $H^{n+1}(N, M_1; \pi(x)) \approx H_{m-n}(N, M_2; \pi(x)) = 0$ since $\pi_q(N, M_2) = 0$ for $q < n$. Hence the obstruction to lifting F_N to $X(x)$ is zero. This completes the proof of 1.8

QED

In section 2 we give some algebra needed in the non-simply connected case, in section 3, an exposition of n-factoizations of maps, in section 4, modifications needed in the non-simply connected cases, in section 5 the manifold with boundary case, and in section 6 a conjecture concerning the k-invariant in passing from $n = [m/2] - 1$ to $n = [m/2] - 1$.

2 Algebraic Preliminaries

Let Λ be a ring, typically $\Lambda = Z[\pi_1(X)]$, and let DGM denote the category of differential graded Λ modules with differential of degree -1. All modules are left modules except when one needs to be a right module, in which case it is left via $\alpha u = u\alpha^{-1}$. If A is a DGM and G is a Λ module, then

$$H_q(A; G) = H_q(G \otimes_\Lambda A)$$

$$H^q(A; G) = H_q(Hom_\Lambda(A, G))$$

$$H_q(A) = H_q(A; \Lambda)$$

$$H^q(A) = H^q(A; \Lambda)$$

Theorem 2.1. *If A is a free DGM and $H_q(A) = 0$ for $q < n$, then the usual map*

$$\Theta : H^q(A; G) \longrightarrow Hom_\Lambda(H_q(A), G)$$

is an isomorphism.

proof. Construct a free DGM B and a map $h : B \longrightarrow A$ such that $h_* : H_*(B) \longrightarrow H_*(A)$ is an isomorphism and $B_q = 0$ for $q < n$. Then h is a DGM homotopy equivalence and the theorem is clearly true for B.

QED

If $f : A \longrightarrow B$ is in DGM, the mapping cone construct $c(f)$, as in section 1,is a DGM.

3 THE MOORE-POSTNIKOV DECOMPOSITION OF A MAP

Suppose X and Y are path connected topological spaces. We define the mapping cylinder of a map $f : X \longrightarrow Y$ by

$$C_f = Y \bigcup (X \times I)/\{(x,1) \ \sim f(x), (x,0) \sim x\}.$$

Sometimes we denote the pair (C_f, X) by (Y, X). Recall a *n-factorization of a map* $f : X \longrightarrow Y$ is a factorization, $f_n : X \longrightarrow X_n$ and $g_n : X_n \longrightarrow Y$ such that $g_n f_n$ is homotopic to f, $\pi_i(X_n, X) = 0$ for $i \leq n+1$ and $\pi_i(Y, X_n) = 0$ for $i >= n+1$. If g_n is fibre maps and $g_n f_n = f$ it is a n-factorization.

The existence if an n-factoization is easily shown by starting with X and attaching cells in dimensoin greater than $n+1$ to produce $g_n : X_n \longrightarrow Y$ such that $(g_n)_* : \pi_i(X_n) \longrightarrow \pi_i(Y)$ is an injection for $i = n+1$ and an isomorphism for $i > n+1$. It can then be converted to a fibration. The Moore-Postnikov decoposition of f provides maps $f_n : X \longrightarrow X_n$, $p_n : X_n \longrightarrow X_{n-1}$ and $g_n : X_n \longrightarrow Y$ such that f_n and g_n give an n-factorization, g_n is a fibration, $g_n = g_{n-1} p_n$ and p_n is a fibration with fiber $K(\pi_{n+1}(Y, X), n+1)$. We construct the Moore-Postnikov decomposition in order to express the k-invariant of p_n in terms of $(f_{n-1})_* : C_*(X) \longrightarrow C_*(X_{n-1})$. To construct X_0 and X_1 we attach cells and convert to fibrations. Note for $n > 0$, $\pi_1(X_n) = \pi_1(X)$

Let $\pi = \pi_1(X)$ and $\Lambda = Z[\pi]$. Rather than dealing with local coefficents, we pass to the universal covers of all the spaces involved, that is, we work in the category $\mathbf{T}_\pi$ of spaces X with free π action and π equivariant maps such that X is 1-connected and X/π has the homotopy type of a CW complex of finite type and $X \longrightarrow X/\pi$ is a covering map. Note $\mathbf{T}_\pi$ is closed under the mapping cylinder construction. Also, the singular chain group, $C_*(X) = C_*(X; Z)$ is a free DGM over Λ and $H_*(X) = H_*(C_*(X)) = H_*(C_*(X); \Lambda))$.

Suppose $f : X \longrightarrow Z$ is in $\mathbf{T}_\pi$, $n > 1$ and f satisfies $\pi_i(Z_f, X) = 0$ for $i \leq n$. Let $\pi_n = H_{n+1}(Z_f, X) = \pi_{n+1}(Z_f, X)$ and $K_{n+1} = K(\pi_n, n+1)$. The π action on π_n induces π action on K_{n+1}. Let $h : (Z_f, X) \longrightarrow (K_{n+1}, 0)$ be a π equivariant map inducing an isomorphism on $\pi_{n+1}(\ )$; 0 is the 0 element and is left fixed by π. Let $j : Z \subset C_f$, $k = hj$ and

$$\bar{Z} = \{(z, \alpha) \in Z \times K_{n+1}^I | \alpha(1) = k(z)\}$$

Let $\bar{p} : \bar{Z} \longrightarrow K_{n+1}$ by $\bar{p}(z, \alpha) = \alpha(0)$. Then $\bar{p}$ is a fibration with fibre

$$Z(k) = \{(z, \alpha) \in Z \times K_{n+1}^I | \alpha(0) = 0 \ and \ \alpha(1) = k(z)\}.$$

Let $\tilde{f} : X \longrightarrow Z(k)$ by $\tilde{f}(x) = (f(x), h(x,))$. Then $p : Z(k) \longrightarrow Z$, where $p(z, \alpha) = z$, is a fibration with fibre $K(\pi_n, n)$ and k-invariant k.

Lemma 3.1. *$\pi_i(Z(k), X) = 0$ for $i \leq n+1$ and $\pi_i(Z, Z(k)) = 0$ for $i > n+1$.*

Proof. : Let $r : Z \subset \bar{Z}$. Note $\bar{p}_* : \pi_*(\bar{Z}, Z(k)) \longrightarrow \pi_*(K_{n+1}, 0)$ is a isomorphism. This together with the homotopy group exact sequence of the triple $(C_{r\tilde{f}}, C_{\tilde{f}}, X)$ shows that to prove the first part of the lemma it is sufficient to show that the inclusion map i induces an isomorphism

$$i_* \pi_{n+1}(C_{r\tilde{f}}, X) \longrightarrow \pi_{n+1}(C_{r\tilde{f}}, C_{\tilde{f}}).$$

Let

$$A : \pi_*(C_{r\tilde{f}}, C_{\tilde{f}}) \longrightarrow \pi_*(\bar{Z}, Z(k)) \longrightarrow \pi_*(K_{n+1}, 0)$$

be the map induced by $C_{r\tilde{f}} \longrightarrow K_{n+1}$ which sends $(z, \alpha) \in \bar{Z}$ to $(z, \alpha) \in \bar{Z}$ and then to $\alpha(0) \in K_{n+1}$ and $(x, t) \in X \times I$ to $r\tilde{f}(x) = (f(x), h(x,))$ and then to $h(x, 0)$. Note A is an isomorphism. Also let

$$B : \pi_*(C_{r\tilde{f}}, X) \longrightarrow \pi_*(C_f, X) \longrightarrow \pi_*(K_{n+1}, 0)$$

be the map which sends (z, α) to z and then to $h(z)$ and sends (x, t) to (x, t) and then to $h(x, t)$. The map B is an isomorphism for $* = n+1$. The maps inducing A and Bi_* are homotopic by the homotopy $F((z, \alpha), s) = \alpha(s)$ and $F((x, t), s) = h(x, st)$. Hence $A = i_* B$ and therefore i_* is an isomorphism for $* = n+1$.

The second part of the lemma is proved as follows: Let $g : (Z_p, Z(k)) \longrightarrow (\bar{Z}, Z(k))$ be the map which sends $((z, \alpha), t)$ to (z, β_t), where $\beta_t = \alpha(t + (1-t)s)$. The map g gives a ladder on homotopy groups and the five lemma shows that g induces an isomorphism on homotopy groups. Then $\pi_*(\bar{Z}, Z(k)) \approx \pi_*(K_{n+1})$ gives the desired result.

QED

Recall $c(f_*)$ is the algebraic mapping cone of f_*. In the above construction of $\tilde{f} : X \longrightarrow Z(k)$ from $f : X \longrightarrow Z$ we replace $C_*(Z_f, X)$ by $c(f_*)$ as follows. Let $B\pi = K(\pi, 1)$, $E\pi \longrightarrow B\pi$ be the universal cover and let $\bar{\eta} : X/\pi \longrightarrow B\pi$ be a map inducing an isomorphism on π_1 and $\eta : X \longrightarrow E\pi$ be an equivariant lifting of $\bar{\eta}$. Let

$$Z \cup_f \hat{X} = Z \bigcup (X \times I) \bigcup E\pi$$

with $(x,0)$ identified to $\eta(x)$ and $(x,1)$ identified to $f(x)$. Note $\mathbf{T}_\pi$ is closed under this mapping cone construction and since $E\pi$ is contractible, $(Z_f, X) \longrightarrow (Z \bigcup_f \hat{X}, \hat{X})$ induces an isomorphism in homology. Also the usual map $c(f_*) \longrightarrow C_*(Z \bigcup_f \hat{X}, \hat{X})$ induces an isomorphism in homology. For a map $k : Z \longrightarrow K_{n+1}$, let $p : Z(k) \longrightarrow Z$ be the fibration

$$Z_k = \{(z,\alpha) \in Z \times K_{n+1}^I | \alpha(0) = 0, \alpha(1) = k(z)\}$$

and $p(z,\alpha) = z$.

Suppose $f : X \longrightarrow Z$ in $\mathbf{T}_\pi$ satisfies $\pi_i(Z,X) = 0$ for $i \leq n$ Then by 2.1 the natural map

$$\theta : H^{n+1}(c(f_*), H_{n+1}(c(f_*)) \longrightarrow Hom_\Lambda(H_{n+1}(c(f_*)), H_{n+1}(c(f_*),))$$

is an isomorphism. Let $j : C_*(Z) \longrightarrow c(f_*)$ be the inclusion. Then Lemma 3.1 yields:

Theorem 3.2. *If $k : Z \longrightarrow K(H_{n+1}(c(f_*)), n+1)$ represents $j^*\theta^{-1}(id)$, then f lifts to $f : X \longrightarrow Z_k$ and for any choice of k and lifting of f, $\pi_i(Z,X) = 0$ for $i \leq n+1$.*

An imediate corollary of Theorem 3.2 is:

Corollary 3.3. *If $f_{n-1} : X \longrightarrow X_{n-1}$ and $g_{n-1} : X_{n-1} \longrightarrow Y$ is an (n-1)-factorization of $f : X \longrightarrow Y$, then the construction in Theorem 3.2 applied to f_{n-1}yields an n-factorization of f, $f_n : X \longrightarrow X_n$ and $g_n = g_{n-1}p$.*

4 The Non-simply Connected Case

Suppose $\xi \longrightarrow X/\pi$ is a vector bundle. Then the Steifel-Whitney class $w_1(\xi)$ gives a homomorphism of $w : \pi \longrightarrow \{1,-1\}$. If G is a Λ module, let $\bar{G}$ be G with its Λ structure given by $\alpha\bar{g} = \bar{\alpha}g$ for $\alpha \in \pi$ and $g \in G$. Let J denote the integers with the trivial Λ action.

As in section 1, let $\bar{C}_q(X;G) = C(^{m-q}(X;\bar{G})$, with differential δ. For $x \in \omega \in H_m(X;\bar{J})$ and $G = \Lambda$, let $P = \cap x : \bar{C}_*(X) = C^{m-*}(X) \longrightarrow C_*(X)$ and $D(X,x) = c(P)$

Definition 4.1. *Suppose X is a DGM. We define an element $\omega \in H_m(X;\bar{J})$ to be a Poincaré n-orientation if*

$$\cap\omega : H^{m-q}(X;\bar{\Lambda}) \longrightarrow H_q(X;\Lambda)$$

is an isomorphism for $m-n \leq q \leq n$, *an injection for* $q = m-n-1$ *and an epimorphism for* $q = n+1$ *,or equivalently, if and only if* $H_q(D(X,x)) = 0$ *for* $m-n \leq q \leq n+1$ *. We define* ω *to be a strong Poincaré n-orientation if in addtion, for* $x \in \omega$ *and* G *a* Λ *module,*

$$\theta : H^{n+1}(D(X,x), G) \longrightarrow Hom_\Lambda(H_{n+1}(D(X,x)), G)$$

is an isomorphism.

Lemma 4.2. *Suppose* Z *is in* $\mathbf{T}_\pi$, $\alpha \in H_m(Z)$ *is a Poincaré m-orientation for* Z *and* $f : Z \longrightarrow X$ *satisfiing* $\pi_i(X,Z) = 0$ *for* $i \leq n$, $2(n-1)+1 \geq m$, *and* $x \in f_*(\alpha)$. *Then* x *is a strong Poincaré n-orientation for* X.

proof. Consider the commutative diagram:

$$\begin{tikzcd}
H^{n+1}(c(f_*); G) \arrow[r, "\Theta"'] \arrow[d, "a"] & Hom_\Lambda(H_{n+1}(c(f_*)), G) \arrow[d, "b"] \\
H^{n+1}(D(X,x); G) \arrow[r, "\Theta"'] & Hom_\Lambda(H_{n+1}(D(X,x), G)
\end{tikzcd}$$

By lemma 3.1 the upper Θ is an isomorphism. The argument given in the proof of 1.7 that

$$H_{n+1}(D(X,x)) \longrightarrow H_{n+1}(c(f_*)$$

is an isomorphism carries over to the present situation showing that b is an isomorphism. Hence it is sufficient to show that a is an isomorphism. The short exact sequence

$$0 \longrightarrow C_*(X) \longrightarrow D(X,x) \longrightarrow \bar{C}_*(X) \longrightarrow 0$$

maps by (i_*, h) to the exact sequence

$$0 \longrightarrow C_*(X) \longrightarrow c(f_*) \longrightarrow \bar{C}_*(Z) \longrightarrow 0$$

giving a commutative ladder when $H^q(\ \ : G)$ is applied. Using the five lemma, it is sufficient to show, when $y \in \alpha$

$$\bar{C}_*(X) \xrightarrow{f^*} \bar{C}_*(Z) \xrightarrow{\cap y} C_*(Z)$$

induces an isomorphism on $H^q(\ ; G)$ for $q = n$ and $n+1$. Since X/π and Z/π have the homotopy type of CW complexes of finite type, $C_*(X)$ and $C_*(Z)$ are DGM homotopy equivanlent to free DGM's finitely generated in each dimension. Thus applying $H^q(\ ; G)$ to the above maps yields:

$$H^q(Z; G) \rightarrow H_{m-q}(Z; G) \rightarrow H_{m-q}(X; G)$$

which are isomorphisms for $q \geq n$.

QED

Corollary 4.3. *The theorems and proofs in section section 1 remain true when "X is simply connected" is replaced by "X is in* $\mathbf{T}_\pi$*".*

5 MANIFOLDS WITH BOUDARY

In this section we assume all our spaces are simply connected and leave to the reader extending the results to the general case. Suppose (X, A) and (Z, B) are CW complex pairs and ζ and ξ are k-plane bundles over X and Z, respectively.

Definition 5.1. *A bundle pair* $(\zeta, \zeta|A)$ *over* (X, A) *is an n-type if* ζ *and* $\zeta|A$ *are n-types. An n-type for a pair* $(\xi, \xi|B)$ *is a map* $F{:}(\xi, \xi|B) \longrightarrow (\zeta, \zeta|A)$ *such that* $F : \xi \longrightarrow \zeta$ *is an n-type for* ξ *and* $F|(\xi|B)$ *is an n-type for* $\xi|B$*.*

Suppose $x \in \omega \in H_m(X : A)$. Let $\bar{C}_q(X) = C^{m-q}(X, A)$, $P : \bar{C}_q(X) \longrightarrow C_q(X)$ by $P(u) = u \cap x$ and $D(X, A, x) = c(P)$.

Definition 5.2. *The chain x above is an n-orientation if* $H_q(D(A, \partial_* x)) = 0$ *and* $H_q(D(X, A, x)) = 0$ *for for* $m - n \leq q \leq n + 1$*.*

Then, as in section 1, We obtain fibrations $X(x) \longrightarrow X$ and $A(\partial_* x) \longrightarrow A$.

Lemma 5.3. *There is a map* $\bar{j} : A(\partial_* x) \longrightarrow X(x)$ *covering the inclusion* $j : A \longrightarrow X$*.*

Proof. That the k=invariant of $X(x)$ goes to zero in A follows from the exactness of

$$H^{n+1}(D(X, A, x)) \longrightarrow H^{n+1}(X) \longrightarrow H_{m-n-1}(X, A),$$

the commutativity of

$$\begin{tikzcd}
H^{n+1}(X) \arrow[r] \arrow[d] & H_{m-n-1}(X, A) \arrow[d] \\
H_{n+1}(A) \arrow[r, "\cap \partial_* x"'] & H_{m-n-2}(A)
\end{tikzcd}$$

and $\cap \partial_* x$ being an injection.

QED

We need $\alpha \in \pi_{m+k}(T(\zeta), T(\zeta|A)$ to choose $\bar{j}$ as indicated in 5.4 below.

Suppose M is a smooth, simply connected, closed compact m-manifold with $(M, \partial M)$ embedded in (R^{m+k}, R^{m+k-1}) such that $\nu_{\partial M} = \nu_M|\partial M$.

The techniques discribed in section 1 yield:

Theorem 5.4. *Suppose* $2(n-1)+1 \geq m$ *and* $F : (\nu_M, \nu_{\partial M}) \longrightarrow (\zeta, \zeta|A)$ *is an* $(n-1)$*-type for* $(\nu_M, \nu_{\partial M})$ *and* $[M] \in H_m(M, \partial M)$ *is the orientation of* M*. If* $x \in (F_M)_*([M])$*, then the map* $\bar{j}$ *(in lemma5.3) can be chosen so as to be an inclusin such that* F *lifts to a map* $F' : (\nu_M, \nu_{\partial M}) \longrightarrow (p^*\zeta, p^*\zeta|A(\partial_* X)$ *. The map* F' *is an* n*-type.*

6 Passing from $n = [m/2] - 1$ to $n = [m/2]$

In this section we make a few observations concerning the passage from an (n-1)-type vector bundle to an n-type when the n-type is the n-type of the normal bundle of a simply connected, closed compact 2n-manifold. We then offer a general conjecture addressing this issue.

The following is a simple exercise in homotopy theory. Suppose π is a finitely generated abelian group, τ is its torsion subgroup and $\lambda : \pi \longrightarrow \pi/\tau$ is the quotient map.

Lemma 6.1. *If* $p : E \longrightarrow X$ *is a fibration induced by* $k : X \longrightarrow K(\pi, n+1)$ *from the contractible fibration over* $K(\pi, n+1)$*,* p *can be factored into fibrations* $E \longrightarrow E' \longrightarrow E$ *where* E' *is induced by* λk *and* E *is induced by* $k' : E' \longrightarrow K(\pi/\tau, n+1)$ *where* $k'|K(\pi/\tau, n)$ *is the Bockstein on the fundamental class in* $H^n(K(\pi/\tau, n); \pi/\tau)$*.*

Applying this decomposition to the base spaces of $\zeta_n \longrightarrow \zeta_{n-1}$ one obtains $\zeta_n \longrightarrow \zeta \longrightarrow \zeta_{n-1}$. We call ζ an $n-1/2$ type, denote it by $\zeta_{n-1/2}$, and define the $n-1/2$ type of a bundle ζ to be a map $\zeta \longrightarrow \zeta_{n-1/2}$ such $\pi_i(X_{n-1/2}, X)$ is 0 for $i < n+1$ and finite for $i = n+1$ and $\pi_i(BO_k, X_{n-1/2})$ is 0 for $i > n+1$ and free abelian for $i = n+1$. The proof of the first part of 1.8 (omitting the uniqueness assertions) goes through with the n in 1.8 replaced by $n + 1/2$ giving:

Theorem 6.2. *Suppose* M *is a simply connected smooth closed, compact 2n-manifold and* $F : \nu_M \longrightarrow \zeta$ *is an* $(n-1/2)$*-type for* ν_M *and* $[M] \in H_m(M)$ *is the orientation of* M*. If* $x \in (F_M)_*([M])$*, then* F_M *lifts to a map* $F'_M : M \longrightarrow X(x)$ *giving a map* $F' : \nu_M \longrightarrow p^*\zeta$ *and the map* F' *is an* n*-type for* ν_M

Again, simple homotopy theory shows that if $\nu_M \longrightarrow \zeta_{n-1/2} \longrightarrow \zeta_{n-1}$ give $n-1/2$ and $n-1$ types for ν_M, then the fibre of $\zeta_{n-1/2} \longrightarrow \zeta_{n-1}$ is a $K(\pi, n)$, where π is the free abelian group $H_{n+1}(X_{n-1}, M)/torsion$ and the k-invariant $k \in H^{n+1}(X_{n-1}; \pi) \approx Hom(\pi^*, H^{n+1}(X_{n-1}))$ corresponds to the natural map $H^{n+1}(X_{n-1}, M) \longrightarrow H^{n+1}(D(x))$, which is an injection, composed with $H^{n+1}(D(x)) \longrightarrow H^{n+1}(X_{n-1})$. ($^* = Hom(\ , Z)$.

Suppose ζ is an n-type over X, $\alpha \in \pi_{m+k}(T(\zeta))$ is such as to give an n-orientation x and the associated Wall invariant vanishes. For $2n+1 \geq m$, one can view (X, ζ, α) as a recipe for building a manifold close to being unique up to a diffeomorphism. When $n = [m/2]-1$ what could substitute for the n-orientation? We offer a conjecture:

Let $L_m(X)$ be the bordism group based on coalgebra maps $h : \mathbf{A} \longrightarrow C_*(X)$ where $\mathbf{A}$ is an m-Poincare coalgeba. Let $W : \Omega_m(\zeta) \longrightarrow L_m(X)$ by sending $M \longrightarrow X$ to $C_*(M) \longrightarrow C_*(X)$.

Conjecture. *Suppose ζ is a $[m/2]-1$ type over X, $h : \mathbf{A} \longrightarrow C_*(X)$, $H_q(c(h)) = 0$ for $q \leq n$ and $\{h\} = W(\alpha)$. As in section 1, h gives a fibration $p : X(h) \longrightarrow X$. Then α lifts to $\alpha' \in \Omega_m(p^*\zeta)$ such that there is an $(M, f) \in \alpha'$ where f is a $[m/2]$ type for ν_M.*

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