Homological Dimension in Noetherian Rings

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$$\alpha_i = \frac{3}{4\pi N} \sum_s^{(i)} A_s. \tag{8}$$

The values of α_i in cubic angstroms are 1.81 for CH_2, 1.82 for C = O, and 1.42 for NH. If we assume that the mean polarizabilities are additive, the mean polarizability per turn, α_1, is 3.69×5.05 Å.[3] The anisotropy ratio β may be taken as about $^1/_3$ for the glycine residue, and the molecular weight per turn is 3.69×57.05. An estimated specific rotation for an infinitely long right-handed alpha helix of polyglycine in an aqueous solution with index of refraction 1.35 therefore is $+132°$.

The available experimental measurements of the optical activity of polypeptides cannot be applied to test the validity of equation (7) because these polypeptides are mixtures of right- and left-handed helices. Some polypeptides chains containing constituents with asymmetric centers may, because of asymmetric synthesis, attain a helical configuration in a preferential direction. However, the extent of this partial resolution, if it exists, is not known.

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[1] J. G. Kirkwood, *J. Chem. Phys.*, **5**, 479, 1937.

[2] D. D. Fitts and J. G. Kirkwood, *J. Am. Chem. Soc.*, **77**, 4940, 1955.

[3] L. Pauling and R. B. Corey, these PROCEEDINGS, **37**, 235, 1951.

[4] F. Eisenlohr, *Z. physik. Chem.*, **75**, 585, 1910; **79**, 129, 1912.

HOMOLOGICAL DIMENSION IN NOETHERIAN RINGS

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1. *Introduction.*—It is our purpose in this note to present some new results in the theory of commutative Noetherian rings (with special emphasis on local rings) which we have obtained using the concepts and techniques of homological algebra, recently introduced by H. Cartan and S. Eilenberg. Since a more detailed account of these and other results will soon appear elsewhere, we take the liberty of omitting most proofs and bibliographical material.

2. *Preliminaries.*—Throughout this note, R will denote a commutative, Noetherian ring. If I is an ideal in R, we denote the rank of I by $\dim I$. We define $\dim R = \sup_I \dim I$, where I runs through all the ideals of R. By the integer $[I]$, we mean the smallest number of elements which generate I. We say that R is a local ring if R has a unique maximal ideal which we shall always denote by m. It is a classical result that $\dim R = \dim m \leq [m]$. R is called a regular local ring if $\dim R = [m]$.

Let M be an R-module. A projective resolution of M is a projective, acyclic complex X over M, i.e., a sequence of R-modules and R-homomorphisms,

$$\ldots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \ldots \to X_1 \xrightarrow{d_1} X_0 \to 0 \tag{2.1}$$

such that each X_n is a direct summand of a free R-module, $d_{n+1}(X_{n+1}) = d_n^{-1}(0)$ for all $n > 0$, and $X_0/d_1(X_1) \cong M$. We define the homological dimension of M, h dim M, to be the smallest integer n such that there is a projective resolution of M with $X_{n+1} = 0$, if such an integer exists. Otherwise we define the homological dimension of M to be ∞.

A basic result concerning h dim M is that if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules, with M projective and M'' not projective, then h dim $M'' = 1 +$ h dim M'.

The global dimension of the ring R, gl dim R, is defined to be the $\sup_M$ h dim M, where M runs through all R-modules. By a result of Auslander, we know that we may confine our attention to finitely generated R-modules.

3. *Codimension.*—Let R be a local ring with maximal ideal m. Cartan and Eilenberg have shown that if M is a finitely generated R-module, then h dim $M \leq$ h dim R/m. Therefore, we have the result that gl dim R = h dim $R/m = 1 +$ h dim m.

3.1. LEMMA. *Let I be an ideal in the local ring R, and X an element in m such that $I{:}(x) = I$, where $I{:}(x) = \{r \in R/r\ x \in I\}$. Then h dim $(I, x) = 1 +$ h dim I.*

Suppose now that R is a regular local ring of dimension n. Then R is an integral domain, and if $m = (u_1, \ldots, u_n)$, $(u_1, \ldots, u_i)$ is a prime ideal for each $i = 1, \ldots, n$. Therefore, from Lemma 3.1, we obtain

3.2. THEOREM. *If R is a regular local ring, then gl dim R = dim R.*

From Lemma 3.1 we can also deduce that m belongs to an ideal I in a regular local ring if and only if r dim I = h dim m.

Let I be a proper ideal in the ring R. We say that $x_1, \ldots, x_p \in R$ is an R/I-sequence if $I{:}(x_1) = I$, $(I, x_1, \ldots, x_i){:}(x_{i+1}) = (I, x_1, \ldots, x_i)$ for all $i = 1, \ldots, p-1$, and $(I, x_1, \ldots, x_p) \neq R$. If R is a local ring, it readily follows that the number of elements in an R/I-sequence is bounded and that $x_1, \ldots, x_p$ is a maximal R/I-sequence if and only if m belongs to $(I, x_1, \ldots, x_p)$. It is trivial to see that every R/I-sequence can be imbedded in a maximal one.

3.3. THEOREM. *If R is a local ring and I is an ideal in R, then all maximal R/I-sequences have the same number of elements.*

Proofs: Suppose that R is a regular local ring and dim $R = n$. Then $x_1, \ldots, x_p$ is a maximal R/I-sequence if and only if m belongs to $(I, x_1, \ldots, x_p)$, or, equivalently, h dim $(I, x_1, \ldots, x_p)$ = h dim m. But, by Lemma 3.1, h dim $(I, x_1, \ldots, x_p) = p +$ h dim I. Thus p = h dim m − h dim I, which proves the invariance of p. It is easy to see that the theorem is also true for all factor rings of regular local rings. If R is an arbitrary local ring, we know that $x_1, \ldots, x_p$ is a maximal R/I-sequence if and only if $x_1, \ldots, x_p$ is a maximal R^*/I^*-sequence, where R^* is the completion of R, and $I^* = R^*I$. Therefore, it suffices to prove the theorem for complete local rings. But every complete local ring is a factor ring of a regular local ring. *Q.E.D.*

An interesting consequence of Theorems 3.2 and 3.3 is that every regular local ring of dimension 2 is a unique factorization domain.

Theorem 3.3 leads us to introduce the following new invariants. We define the *codimension of R/I* (codim R/I), where I is an ideal in the local ring R, to be the number of elements in a maximal R/I-sequence. We can show that codim R =

$\sup_M$ h dim M, where M ranges over all finitely generated R-modules of finite homological dimension.

Note: Let R be a ring, and M an R-module. We define an M-sequence to be a sequence $x_1, \ldots, x_p$ of elements in R such that x_1 is not a zero division for M, and x_{i+1} is not a zero division for $M/(x_1, \ldots, x_i)M$, for all $i = 1, \ldots, p - 1$. This is a direct generalization of an R/I-sequence.

4. *Regular Local Rings.*—Let R be a ring and S a multiplicatively closed subset of R not containing 0, and let R_S be the ring of quotients of R with respect to S. Since $\mathrm{Tor}_n^R (R_S, M) = 0$, for all $n > 0$, we have h $\dim_R M \geq$ h $\dim_{R_S} R_S \otimes_R M$. If I is an ideal in R_S, then $R_S/I \approx R_S \otimes_R R/R \cap I$. Thus we have gl dim $R \geq$ gl dim R_S. Therefore, if R is a regular local ring and P is a prime ideal in R, then gl dim R_P is finite. It is well known that for regular geometric local rings, as well as for regular, nonramified complete local rings, R_P is also a regular local ring for every prime P. This observation, together with some direct computations, led the authors to conjecture

4.1. THEOREM. *A local ring R is regular if and only if gl dim R is finite.*

The proof of this theorem is a direct result of the following lemmas, the first of which was established by the authors and the second of which is an immediate consequence of a result of Serre.

4.2. LEMMA. *If R is a local ring of finite homological dimensions, then gl dim R = dim R.*

4.3. LEMMA. *If R is a local ring with maximal ideal m, then h dim $R/m \geq [m]$.*

4.4. COROLLARY. *If R is a regular local ring, then so is R_P for every prime ideal P in R.*

The last result leads one to say that a ring R is *regular* if and only if R_P is a regular local ring for all prime ideals P of R. For a regular ring R, we have gl dim R = dim R. If gl dim R is finite, it is clear that R is regular. Also, it is not difficult to show that all regular rings are finite direct sums of regular integral domains.

An ideal I in a regular ring R is said to be *perfect* if h dim R/I = dim I. Since h $\dim_R R/I \geqslant \sup_P \dim P$, where P ranges over all the prime ideals belonging to I, a perfect ideal is unmixed. We are able to establish the following version of the Cohen-MacAulay theorem:

4.5. THEOREM. *If R is a regular ring, and $I = (x_1, \ldots, x_n)$ is an ideal of rank n, then the $x_1, \ldots, x_n$ are an R-sequence regardless of the order in which they are written. Consequently, I is a perfect ideal and is therefore unmixed.*

It can also be shown that if dim $R \leqslant 2$, and R is a regular domain, then every minimal prime ideal in R is invertible.

We conclude this note with

4.6. THEOREM. *If R is a Zariski ring and M is an R-module, then h $\dim_R M \geq$ h $\dim_{R^*} M^*$,* where *R^* is the completion of R and M^* the completion of M. If R is a local ring, then the above inequality becomes an equality and gl. dim R = gl. dim R^*.*

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