



## Corrections to Codimension and Multiplicity

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*Corrections to*  
**Codimension and Multiplicity**

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**1. Errata and corrections**

1. In proving Proposition 3.5 on page 637, we assumed tacitly that  $R$  has finite length. The statement of the proposition is correct as given, i.e., without the additional hypothesis that  $R$  has finite length. A proof for this more general situation can be obtained merely by applying induction on  $s$  and Theorem 3.3 immediately following the first paragraph of the proof of 3.5.

2. The second paragraph of Theorem 5.6 on page 646 should read as follows :

Let  $E$  be an  $R$ -module having the following properties :

(a)  $\dim E = \dim R$ .

(b) For every set of elements  $x_1, \dots, x_k$  in  $m$  such that  $\text{rank}(E/(x_1, \dots, x_k)E) = k$ , the submodule (0) in  $E/(x_1, \dots, x_k)E$  is unmixed.

If a set of elements  $z_1, \dots, z_s$  in  $m$  ( $s = \dim R$ ) can be found such that  $\text{rank}(E/(z_1, \dots, z_k)E) = k$  for  $1 \leq k \leq s$ , then  $E$  is a Macaulay  $R$ -module.

The proof of this statement is supplied by lines 3 to 31 on page 647, if we let  $z_1 = x$ . That  $E/xE$  and  $R/(x)$  satisfy the additional condition above is easily seen.

It should be remarked that if  $E=R$ , then the existence of the elements  $z_1, \dots, z_s$  need not be postulated. Moreover, the existence of  $z_1, \dots, z_s$  need not be postulated if one assumes that all the maximal ideals of  $R$  have the same rank (e.g., if  $R$  is a local ring).

3. On page 651, the conclusion of Theorem 6.5 should read: "... for all  $i \geq n$  and so  $\text{codim} E > s - n$ ".

**2. A non-spectral sequence proof of Theorem 3.6**

**THEOREM.** *Let  $E$  be a finitely generated module over the (arbitrary) noetherian ring  $R$ , and let  $\mathfrak{b} = (x_1, \dots, x_s)$  be an ideal in  $R$  such that  $E/\mathfrak{b}E$  has finite length. Then  $\chi(H(E_{x_1, \dots, x_s})) = \Delta^s \chi(n; E, \mathfrak{b})$  for  $n$  sufficiently large.*

**PROOF.** We let  $C = E_{x_1, \dots, x_s}$  and  $\bar{C} = G(E; \mathfrak{b})_{x_1, \dots, x_s}$  and we still see that  $H_i(C)$  and  $H_i(\bar{C})$  have finite length for all  $i \geq 0$ . Since  $H_i(\bar{C}) =$

$\sum_k H_i^{(k)}$ , there is an integer  $n_0$  such that  $H_i^{(k)} = 0$  for all  $k \geq n_0$  and all  $i$ .  
 Let  $C^{(k)}$  be the complex :

$$0 \rightarrow \mathfrak{b}^k C_s \rightarrow \mathfrak{b}^{k+1} C_{s-1} \rightarrow \dots \rightarrow \mathfrak{b}^{k+s} C_0 \rightarrow 0$$

with the differentiation induced by that of  $C$ . Then  $C^{(k)}$  is a subcomplex of  $C$  and we have the exact sequence

$$0 \rightarrow C^{(k)} \rightarrow C \rightarrow C/C^{(k)} \rightarrow 0 .$$

The aim of the remaining argument is to show that  $H_i(C^{(k)}) = 0$  for all  $i \geq 0$  and for  $k$  sufficiently large. If we show this, then we will have  $H_i(C) \approx H_i(C/C^{(k)})$  so that  $\sum (-1)^i L(H_i(C)) = \sum (-1)^i L(H_i(C/C^{(k)}))$ . However, since  $(C/C^{(k)})_i = C_i/\mathfrak{b}^{k+s-i} C_i$  has finite length for each  $i$ , we will then have  $\sum (-1)^i L(H_i(C/C^{(k)})) = \sum (-1)^i L((C/C^{(k)}))$ , and this last sum is easily seen to be equal to  $\Delta^s \chi(n; E, \mathfrak{b})$  for  $n$  sufficiently large.

To show that  $H_i(C^{(k)}) = 0$ , we proceed as follows. We suppose that  $k > n_0$ . Since  $H_i^{(k)}(\bar{C}) = 0$  for  $k > n_0$ , we have the exactness of

$$(*) \quad 0 \rightarrow \mathfrak{b}^k C_s / \mathfrak{b}^{k+1} C_s \rightarrow \mathfrak{b}^{k+1} C_{s-1} / \mathfrak{b}^{k+2} C_{s-1} \rightarrow \dots \rightarrow \mathfrak{b}^{k+s} C_0 / \mathfrak{b}^{k+s+1} C_0 \rightarrow 0 .$$

Let  $z \in \mathfrak{b}^{k+s-i} C_i$  be an  $i$ -dimensional cycle of  $C^{(k)}$ . Then  $dz = 0$  so that if we let  $\bar{z}$  be the coset of  $z$  in  $\mathfrak{b}^{k+s-i} C_i / \mathfrak{b}^{k+s-i+1} C_i$ , we have that  $\bar{z}$  is a cycle of  $\bar{C}$ . From the exactness of  $(*)$  we can deduce that  $z = z_1 + d(b_1)$  where  $z_1 \in \mathfrak{b}^{k+s-i+1} C_i$  and  $b_1 \in \mathfrak{b}^{k+s-i-1} C_{i+1}$ . Since  $dz = 0$ , we have  $dz_1 = 0$ , so that repeating the argument (using  $k + 1$  in place of  $k$ ), we have  $z = z_2 + d(b_2)$  where  $z_2 \in \mathfrak{b}^{k+s-i+2} C_i$  and  $b_2 \in \mathfrak{b}^{k+s-i-1} C_{i+1}$ . Proceeding in this way, we see that  $z \in \bigcap_{p \geq k+s-i} \mathfrak{b}^p C_i + d(\mathfrak{b}^{k+s-(i+1)} C_{i+1}) = \bigcap_p \mathfrak{b}^p C_i + d(\mathfrak{b}^{k+s-(i+1)} C_{i+1})$ . Therefore,  $Z_i = Z_i(C^{(k)})$  is contained in  $\bigcap_p \mathfrak{b}^p C_i + B_i$  (where  $B_i = d(C_{i+1}^{(k)})$ ). Since  $\bigcap_p \mathfrak{b}^p (C_i/B_i) = \bigcap_p (\mathfrak{b}^p C_i + B_i/B_i)$ , and since the ring and all modules are noetherian, we know that there exists a  $b \in \mathfrak{b}$  such that  $b\bar{c} = \bar{c}$  for all  $\bar{c}$  in  $\bigcap_p \mathfrak{b}^p (C_i/B_i)$  and so, in particular, for all  $\bar{c}$  in  $Z_i/B_i$  (because  $B_i \subset Z_i \subset \mathfrak{b}^p C_i + B_i$ ). Since  $b^t \bar{c} = \bar{c}$  for all positive integers  $t$ , we conclude that for each positive integer  $t$ , there exists a  $b_t$  in  $\mathfrak{b}^t$  such that  $(1-b)H_i(C^{(k)}) = 0$ . (Of course  $b_t$  depends upon  $t$  and  $k$ .) If we let  $(1-b) = (1-b_0) \dots (1-b_s)$ , we have  $b$  in  $\mathfrak{b}^t$  and  $(1-b)H_i(C^{(k)}) = 0$  for  $i = 0, \dots, s$ .

Observe next that since  $\mathfrak{b}H(C) = 0$ , we have  $\mathfrak{b}^t H(C) = 0$  for all  $t$ . Also, since  $\mathfrak{b}^t \cdot C/C^{(k)} = 0$  for  $t \geq k+s$ , we have  $\mathfrak{b}^t H(C/C^{(k)}) = 0$  for  $t \geq k+s$ . Let  $b$  in  $\mathfrak{b}^{k+s}$  be such that  $(1-b)H_i(C^{(k)}) = 0$  for all  $i = 0, \dots, s$ . Then we have the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \rightarrow & H_s(C^{(k)}) & \rightarrow & H_s(C) & \rightarrow & \dots & \rightarrow & H_1(C/C^{(k)}) & \rightarrow & H_0(C^{(k)}) & \rightarrow & H_0(C) & \rightarrow & H_0(C/C^{(k)}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & H_s(C^{(k)}) & \rightarrow & H_s(C) & \rightarrow & \dots & \rightarrow & H_1(C/C^{(k)}) & \rightarrow & H_0(C^{(k)}) & \rightarrow & H_0(C) & \rightarrow & H_0(C/C^{(k)}) & \rightarrow & 0 \end{array}$$

with exact rows and where the vertical maps are multiplication by  $1 - b$ . Since multiplication by  $1 - b$  is the identity map on  $H(C)$  and  $H(C/C^{(k)})$ , and the zero map on  $H(C^{(k)})$ , we may deduce from the “five lemma” that  $H(C^{(k)}) = 0$ , and the proof concludes as indicated above.

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