Annals of Mathematics
Vol. 70 No. 2, September, 1959
Printed in Japan

Corrections to

Codimension and Multiplicity

These Annals, Vol. 68 (1958), 625-657.

By Maurice Auslander and David A. Buchsbaum

(Received May 4, 1959)

1. Errata and corrections

1. In proving Proposition 3.5 on page 637, we assumed tacitly that R has finite length. The statement of the proposition is correct as given, i.e., without the additional hypothesis that R has finite length. A proof for this more general situation can be obtained merely by applying induction on s and Theorem 3.3 immediately following the first paragraph of the proof of 3.5.

2. The second paragraph of Theorem 5.6 on page 646 should read as follows:

Let E be an R-module having the following properties:

(a) $\dim E = \dim R$.

(b) For every set of elements $x_1, \cdots, x_k$ in m such that $\operatorname{rank}(E/(x_1, \cdots, x_k E) = k$, the submodule (0) in $E/(x_1, \cdots, x_k)E$ is unmixed.

If a set of elements $z_1, \cdots, z_s$ in $\mathfrak{m}$ ($s = \dim R$) can be found such that $(E/(z_1, \cdots, z_k)E) = k$ for $1 \leqq k \leqq s$, then E is a Macaulay R-module.

The proof of this statement is supplied by lines 3 to 31 on page 647, if we let $z_1 = x$. That E/xE and $R/(x)$ satisfy the additional condition above is easily seen.

It should be remarked that if $E=R$, then the existence of the elements $z_1, \cdots, z_s$ need not be postulated. Moreover, the existence of $z_1, \cdots, z_s$ need not be postulated if one assumes that all the maximal ideals of R have the same rank (e.g., if R is a local ring).

3. On page 651, the conclusion of Theorem 6.5 should read: "$\cdots$ for all $i \geqq n$ and so $\operatorname{codim} E > s - n$".

2. A non-spectral sequence proof of Theorem 3.6

THEOREM. *Let E be a finitely generated module over the (arbitrary) noetherian ring R, and let $\mathfrak{b}=(x_1, \cdots, x_s)$ be an ideal in R such that $E/\mathfrak{b}E$ has finite length. Then $\chi(H(E_{x:1,\ldots,s})) = \Delta^s\chi(n; E, \mathfrak{b})$ for n sufficiently large.*

PROOF. We let $C = E_{x:1,\cdots,s}$ and $\overline{C} = G(E; \mathfrak{b})_{\bar{x}:1,\cdots,s}$ and we still see that $H_i(C)$ and $H_i(\overline{C})$ have finite length for all $i \geqq 0$. Since $H_i(\overline{C}) =$

$\sum_k H_i^{(k)}$, there is an integer n_0 such that $H_i^{(k)} = 0$ for all $k \geqq n_0$ and all i.

Let $C^{(k)}$ be the complex:

$$0 \to \mathfrak{b}^k C_s \to \mathfrak{b}^{k+1} C_{s-1} \to \cdots \to \mathfrak{b}^{k+s} C_0 \to 0$$

with the differentiation induced by that of C. Then $C^{(k)}$ is a subcomplex of C and we have the exact sequence

$$0 \to C^{(k)} \to C \to C/C^{(k)} \to 0 .$$

The aim of the remaining argument is to show that $H_i(C^{(k)}) = 0$ for all $i \geqq 0$ and for k sufficiently large. If we show this, then we will have $H_i(C) \approx H_i(C/C^{(k)})$ so that $\sum (-1)^i L(H_i(C)) = \sum (-1)^i L(H_i(C/C^{(k)}))$. However, since $(C/C^{(k)})_i = C_i/\mathfrak{b}^{k+s-i} C_i$ has finite length for each i, we will then have $\sum (-1)^i L(H_i(C/C^{(k)})) = \sum (-1)^i L((C/C^{(k)}))$, and this last sum is easily seen to be equal to $\Delta^s \chi(n ; E, \mathfrak{b})$ for n sufficiently large.

To show that $H_i(C^{(k)}) = 0$, we proceed as follows. We suppose that $k > n_0$. Since $H_i^{(k)}(\bar{C}) = 0$ for $k > n_0$, we have the exactness of

$$(*) \qquad 0 \to \mathfrak{b}^k C_s/\mathfrak{b}^{k+1} C_s \to \mathfrak{b}^{k+1} C_{s-1}/\mathfrak{b}^{k+2} C_{s-1} \to \cdots \to \mathfrak{b}^{k+s} C_0/\mathfrak{b}^{k+s+1} C_0 \to 0 .$$

Let $z \in \mathfrak{b}^{k+s-1} C_i$ be an i-dimensional cycle of $C^{(k)}$. Then $dz = 0$ so that if we let $\bar{z}$ be the coset of z in $\mathfrak{b}^{k+s-i} C_i/\mathfrak{b}^{k+s-i+1} C_i$, we have that $\bar{z}$ is a cycle of $\bar{C}$. From the exactness of (*) we can deduce that $z = z_1 + d(b_1)$ where $z_1 \in \mathfrak{b}^{k+s-i+1} C_i$ and $b_1 \in \mathfrak{b}^{k+s-i-1} C_{i+1}$. Since $dz = 0$, we have $dz_1 = 0$, so that repeating the argument (using $k+1$ in place of k), we have $z = z_2 + d(b_2)$ where $z_2 \in \mathfrak{b}^{k+s-i+2} C_i$ and $b_2 \in \mathfrak{b}^{k+s-i-1} C_{i+1}$. Proceeding in this way, we see that $z \in \bigcap_{p \geqq k+s-i} \mathfrak{b}^p C_i + d(\mathfrak{b}^{k+s-(i+1)} C_{i+1}) = \bigcap_p \mathfrak{b}^p C_i + d(\mathfrak{b}^{k+s-(i+1)} C_{i+1})$. Therefore, $Z_i = Z_i(C^{(k)})$ is contained in $\bigcap_p \mathfrak{b}^p C_i + B_i$ (where $B_i = d(C_{i+1}^{(k)})$). Since $\bigcap_p \mathfrak{b}^p (C_i/B_i) = \bigcap_p (\mathfrak{b}^p C_i + B_i/B_i)$, and since the ring and all modules are noetherian, we know that there exists a $b \in \mathfrak{b}$ such that $b\bar{c} = \bar{c}$ for all $\bar{c}$ in $\bigcap_p \mathfrak{b}^p (C_i/B_i)$ and so, in particular, for all $\bar{c}$ in Z_i/B_i (because $B_i \subset Z_i \subset \mathfrak{b}^p C_i + B_i$). Since $b^t \bar{c} = \bar{c}$ for all positive integers t, we conclude that for each positive integer t, there exists a b_i in $\mathfrak{b}^t$ such that $(1-b)H_i(C^{(k)}) = 0$. (Of course b_i depends upon t and k.) If we let $(1-b) = (1-b_0) \cdots (1-b_s)$, we have b in $\mathfrak{b}^t$ and $(1-b)H_i(C^{(k)}) = 0$ for $i = 0, \cdots, s$.

Observe next that since $\mathfrak{b}H(C) = 0$, we have $\mathfrak{b}^t H(C) = 0$ for all t. Also, since $\mathfrak{b}^t \cdot C/C^{(k)} = 0$ for $t \geqq k+s$, we have $\mathfrak{b}^t H(C/C^{(k)}) = 0$ for $t \geqq k+s$. Let b in $\mathfrak{b}^{k+s}$ be such that $(1-b) H_i(C^{(k)}) = 0$ for all $i = 0, \cdots, s$. Then we have the commutative diagram

$$\begin{array}{ccccccccccccccc}
0 \to & H_s(C^{(k)}) & \to & H_s(C) & \to \cdots \to & H_1(C/C^{(k)}) & \to & H_0(C^{(k)}) & \to & H_0(C) & \to & H_0(C/C^{(k)}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H_s(C^{(k)}) & \to & H_s(C) & \to \cdots \to & H_1(C/C^{(k)}) & \to & H_0(C^{(k)}) & \to & H_0(C) & \to & H_0(C/C^{(k)}) & \to 0
\end{array}$$

with exact rows and where the vertical maps are multiplication by $1 - b$. Since multiplication by $1 - b$ is the identity map on $H(C)$ and $H(C/C^{(k)})$, and the zero map on $H(C^{(k)})$, we may deduce from the "five lemma" that $H(C^{(k)}) = 0$, and the proof concludes as indicated above.

BROWN UNIVERSITY