UNIQUE FACTORIZATION IN REGULAR LOCAL RINGS

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In this note we prove that every regular local ring of dimension 3 is a unique factorization domain. Nagata[4] showed (Proposition 11) that if every regular local ring of dimension 3 is a unique factorization domain, then every regular local ring has unique factorization.* Thus, combining these results we have that every regular local ring is a unique factorization domain.

Throughout this note R is a local ring with maximal ideal $\mathfrak{M}$. The definitions and notation follow those of Auslander and Buchsbaum.[1]

PROPOSITION 1. *Let x be in $\mathfrak{M}$ and y in R such that $\mathfrak{a} = (x):y$ satisfies the following conditions: (a) $hd\ \mathfrak{a} \leq 1$ and (b) x is not in $\mathfrak{M}\mathfrak{a}$. Then $\mathfrak{a} = (x)$ and x is not a zero divisor.*

Proof: Suppose x does not generate $\mathfrak{a}$. Since x is not in $\mathfrak{M}\mathfrak{a}$, there exist $a_1, \ldots, a_n$ in $\mathfrak{a}$ ($n > 0$) such that $x, a_1, \ldots, a_n$ form a minimal generating set for $\mathfrak{a}$. Let R^{n+1} denote the direct sum of $n + 1$ copies of R, define $f\colon R^{n+1} \to \mathfrak{a}$ by $f(r_0, \ldots, r_n) = r_0x + \sum_{i=1}^{n} r_ia_i$, and let $K = \operatorname{Ker} f$. Since $x, a_1, \ldots, a_n$ is a minimal generating system for $\mathfrak{a}$, we have that f is an epimorphism and K is contained in $\mathfrak{M}R^{n+1}$. From the exact sequence $0 \to K \to R^{n+1} \to \mathfrak{a} \to 0$ and the fact that $hd\ \mathfrak{a} \leq 1$, we have that K is R-free. Since $\mathfrak{a}$ is not principal, we have that $hd\ \mathfrak{a} = 1$ and thus $K \neq 0$.

Let $N_i = (t_{i0}, \ldots, t_{in})$ be a free basis for K over R ($i = 1, \ldots, m$). Since a_1 is in $\mathfrak{a}$, we have that $ya_1 = -vx$ for some v in R. Let $V = (v, y, 0, \ldots, 0)$ and $T = (a_1, -x, 0, \ldots, 0)$. Then V and T are in K. Let $V = \sum_{j=1}^{m} r_jN_j$ and $T = \sum_{j=1}^{m} s_jN_j$. Now $xV = -yT$. Therefore we have that $\sum_{j=1}^{m} xr_jN_j = \sum_{j=1}^{m} -ys_jN_j$. Since the $\{N_j\}$ are a free basis for K over R, we have that $xr_j = -ys_j$ for all $j = 1, \ldots, m$. But $(x):y = \mathfrak{a}$. Therefore we have that each s_j is in $\mathfrak{a}$. Since $T = (a_1, -x, 0, \ldots, 0) = \sum_{j=1}^{m} s_jN_j$, it follows that $-x = \sum_{j=1}^{m} s_jt_{j1}$. Therefore x is in $\mathfrak{M}\mathfrak{a}$ (since K is contained in $\mathfrak{M}R^{n+1}$) which contradicts the fact that x is not in $\mathfrak{M}\mathfrak{a}$. Thus $\mathfrak{a} = (x)$. The fact that x is not a zero divisor follows from the assumption $hd\ \mathfrak{a} \leq 1 < \infty$ and [2, corollary 6.3].

COROLLARY 2. *Suppose $\mathfrak{p}$ is a prime ideal in R such that $\dim R_{\mathfrak{p}} = 1$ and $hd\ R/\mathfrak{p} \leq 2$. Then $\mathfrak{p}$ is a principal ideal.*

Proof: Since $hd_RR/\mathfrak{p} \geq hd_{R_\mathfrak{p}}R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = gl.\dim R_{\mathfrak{p}}$, it follows that the $gl.\dim R_{\mathfrak{p}}$ is finite. (See reference 1, 1.6 and reference 3; VIII, 2.6[1].) Therefore $R_{\mathfrak{p}}$ is a regular local ring of dimension 1 (see reference 1, 1.9). Let $x_1, \ldots, x_t$ be a min. generating set for $\mathfrak{p}$. Then the $x_1, \ldots, x_t$ considered as elements in $R_{\mathfrak{p}}$ generate $\mathfrak{p}R_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ is a regular local ring of dimension 1, we have that $\mathfrak{p}R_{\mathfrak{p}} = x_jR_{\mathfrak{p}}$ for some j. Let $(x_j) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_l$ be a normal, primary decomposition for (x_j). From $x_jR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ it follows that one of the $\mathfrak{q}_i$ is $\mathfrak{p}$. Let us say $\mathfrak{q}_1 = \mathfrak{p}$. Then for

y in $(\mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_l) - \mathfrak{p}$ we have that $(x_j): y = \mathfrak{p}$. Since x_j is not in $\mathfrak{M}\mathfrak{p}$ and $hd\ \mathfrak{p} \leq 1$, it follows from the previous proposition that $\mathfrak{p} = (x_j)$.

THEOREM 3. *Let R be a local domain of dimension* ≤ 3 *such that* $hd\ R/\mathfrak{p} < \infty$ *for all minimal prime ideals* $\mathfrak{p}$. *Then R is a unique factorization domain.*

Proof: Since R is a noetherian domain, it follows from reference 4; Lemma 1, pg. 408, that it suffices to show that each minimal prime ideal is principal in order to show that R is a unique factorization domain. But by Corollary 2, it will follow that a minimal prime ideal $\mathfrak{p}$ is principal if we can show that $hd\ R/p \leq 2$. Since $hd\ R/\mathfrak{p} < \infty$ we have by reference 1; 3.7 and 1.3 that $hd\ R/\mathfrak{p} + \text{Codim } R/\mathfrak{p} = \text{Codim } R \leq \dim R$. But $\text{Codim } R/\mathfrak{p} \geq 1$ and $\dim R \leq 3$. Thus $hd\ R/\mathfrak{p} \leq 2$, which completes the proof.

Since every module has finite homological dimension over a regular local ring, we have established

COROLLARY 4. *Every regular local ring of dimension* ≤ 3 *is a unique factorization domain.*

THEOREM 5. *Every regular local ring is a unique factorization domain.*

* Prior to this result, Zariski proved that if every complete regular local ring of dimension 3 is a unique factorization domain, then every complete regular local ring is a unique factorization domain (unpublished). Combining this with Mori's and Krull's result that a local ring is a unique factorization domain if it's completion is a unique factorization domain, we obtain another proof of this reduction theorem.

[1] Auslander, M., and D. A. Buchsbaum, "Homological Dimension in Noetherian Rings," *Trans. Am. Math. Soc.*, **85**, 390–405 (1957).

[2] Auslander, M., and D. Buchsbaum, "Codimension and Multiplicity," *Ann. Math.*, **68**, 625–657 (1958).

[3] Cartan, H., and S. Eilenberg, *Homological Algebra* (Princeton, 1955).

[4] Nagata, M., "A General Theory of Algebraic Geometry over Dedekind Rings II," *Am. J. Math.*, **80**, 382–420 (1958).

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*ON THE FALSITY OF EULER'S CONJECTURE ABOUT THE NON-EXISTENCE OF TWO ORTHOGONAL LATIN SQUARES OF ORDER 4t + 2**

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1. Introduction.—The purpose of this paper is to prove a general theorem on the existence of pairwise orthogonal Latin squares (p.o.l.s.) of a given order and to give a counter example to Euler's conjecture[3] that there do not exist two p.o.l.s. of order $4t + 2$.

2. Definitions.—An arrangement of v objects (called treatments) in b sets (called blocks) will be called a pairwise balanced design of index unity and type $(v;\ k_1, k_2, \ldots, k_m)$ if each block contains either $k_1, k_2, \ldots,$ or k_m treatments which are all distinct $(k_i \leq v,\ k_i \neq k_j)$, and every pair of distinct treatments occurs exactly in one block of the design. If the number of blocks containing k_i treatments is b_i, then clearly