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UNIQUE FACTORIZATION IN REGULAR LOCAL RINGS

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In this note we prove that every regular local ring of dimension 3 is a unique factorization domain. Nagata⁴ showed (Proposition 11) that if every regular local ring of dimension 3 is a unique factorization domain, then every regular local ring has unique factorization.* Thus, combining these results we have that every regular local ring is a unique factorization domain.

Throughout this note R is a local ring with maximal ideal \mathfrak{M} . The definitions and notation follow those of Auslander and Buchsbaum.¹

PROPOSITION 1. *Let x be in \mathfrak{M} and y in R such that $\mathfrak{a} = (x) : y$ satisfies the following conditions: (a) $hd \mathfrak{a} \leq 1$ and (b) x is not in $\mathfrak{M}\mathfrak{a}$. Then $\mathfrak{a} = (x)$ and x is not a zero divisor.*

Proof: Suppose x does not generate \mathfrak{a} . Since x is not in $\mathfrak{M}\mathfrak{a}$, there exist a_1, \dots, a_n in \mathfrak{a} ($n > 0$) such that x, a_1, \dots, a_n form a minimal generating set for \mathfrak{a} . Let R^{n+1} denote the direct sum of $n + 1$ copies of R , define $f: R^{n+1} \rightarrow \mathfrak{a}$ by $f(r_0, \dots, r_n) = r_0x + \sum_{i=1}^n r_i a_i$, and let $K = \text{Ker } f$. Since x, a_1, \dots, a_n is a minimal generating system for \mathfrak{a} , we have that f is an epimorphism and K is contained in $\mathfrak{M}R^{n+1}$. From the exact sequence $0 \rightarrow K \rightarrow R^{n+1} \rightarrow \mathfrak{a} \rightarrow 0$ and the fact that $hd \mathfrak{a} \leq 1$, we have that K is R -free. Since \mathfrak{a} is not principal, we have that $hd \mathfrak{a} = 1$ and thus $K \neq 0$.

Let $N_i = (t_{i0}, \dots, t_{in})$ be a free basis for K over R ($i = 1, \dots, m$). Since a_1 is in \mathfrak{a} , we have that $ya_1 = -vx$ for some v in R . Let $V = (v, y, 0, \dots, 0)$ and $T = (a_1, -x, 0, \dots, 0)$. Then V and T are in K . Let $V = \sum_{j=1}^m r_j N_j$ and $T = \sum_{j=1}^m s_j N_j$. Now $xV = -yT$. Therefore we have that $\sum_{j=1}^m x r_j N_j = \sum_{j=1}^m -y s_j N_j$. Since the $\{N_j\}$ are a free basis for K over R , we have that $xr_j = -ys_j$ for all $j = 1, \dots, m$. But $(x) : y = \mathfrak{a}$. Therefore we have that each s_j is in \mathfrak{a} . Since $T = (a_1, -x, 0, \dots, 0) = \sum_{j=1}^m s_j N_j$, it follows that $-x = \sum_{j=1}^m s_j t_{j1}$. Therefore x is in $\mathfrak{M}\mathfrak{a}$ (since K is contained in $\mathfrak{M}R^{n+1}$) which contradicts the fact that x is not in $\mathfrak{M}\mathfrak{a}$. Thus $\mathfrak{a} = (x)$. The fact that x is not a zero divisor follows from the assumption $hd \mathfrak{a} \leq 1 < \infty$ and [2, corollary 6.3].

COROLLARY 2. *Suppose \mathfrak{p} is a prime ideal in R such that $\dim R_{\mathfrak{p}} = 1$ and $hd R/\mathfrak{p} \leq 2$. Then \mathfrak{p} is a principal ideal.*

Proof: Since $hd_R R/\mathfrak{p} \geq hd_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = gl. \dim R_{\mathfrak{p}}$, it follows that the $gl. \dim R_{\mathfrak{p}}$ is finite. (See reference 1, 1.6 and reference 3; VIII, 2.6¹.) Therefore $R_{\mathfrak{p}}$ is a regular local ring of dimension 1 (see reference 1, 1.9). Let x_1, \dots, x_t be a min. generating set for \mathfrak{p} . Then the x_1, \dots, x_t considered as elements in $R_{\mathfrak{p}}$ generate $\mathfrak{p}R_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ is a regular local ring of dimension 1, we have that $\mathfrak{p}R_{\mathfrak{p}} = x_j R_{\mathfrak{p}}$ for some j . Let $(x_j) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_i$ be a normal, primary decomposition for (x_j) . From $x_j R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ it follows that one of the \mathfrak{q}_i is \mathfrak{p} . Let us say $\mathfrak{q}_1 = \mathfrak{p}$. Then for

y in $(q_2 \cap \dots \cap q_t) - \mathfrak{p}$ we have that $(x_j): y = \mathfrak{p}$. Since x_j is not in $\mathfrak{M}\mathfrak{p}$ and $hd \mathfrak{p} \leq 1$, it follows from the previous proposition that $\mathfrak{p} = (x_j)$.

THEOREM 3. *Let R be a local domain of dimension ≤ 3 such that $hd R/\mathfrak{p} < \infty$ for all minimal prime ideals \mathfrak{p} . Then R is a unique factorization domain.*

Proof: Since R is a noetherian domain, it follows from reference 4; Lemma 1, pg. 408, that it suffices to show that each minimal prime ideal is principal in order to show that R is a unique factorization domain. But by Corollary 2, it will follow that a minimal prime ideal \mathfrak{p} is principal if we can show that $hd R/\mathfrak{p} \leq 2$. Since $hd R/\mathfrak{p} < \infty$ we have by reference 1; 3.7 and 1.3 that $hd R/\mathfrak{p} + \text{Codim } R/\mathfrak{p} = \text{Codim } R \leq \dim R$. But $\text{Codim } R/\mathfrak{p} \geq 1$ and $\dim R \leq 3$. Thus $hd R/\mathfrak{p} \leq 2$, which completes the proof.

Since every module has finite homological dimension over a regular local ring, we have established

COROLLARY 4. *Every regular local ring of dimension ≤ 3 is a unique factorization domain.*

THEOREM 5. *Every regular local ring is a unique factorization domain.*

* Prior to this result, Zariski proved that if every complete regular local ring of dimension 3 is a unique factorization domain, then every complete regular local ring is a unique factorization domain (unpublished). Combining this with Mori's and Krull's result that a local ring is a unique factorization domain if its completion is a unique factorization domain, we obtain another proof of this reduction theorem.

¹ Auslander, M., and D. A. Buchsbaum, "Homological Dimension in Noetherian Rings," *Trans. Am. Math. Soc.*, **85**, 390-405 (1957).

² Auslander, M., and D. Buchsbaum, "Codimension and Multiplicity," *Ann. Math.*, **68**, 625-657 (1958).

³ Cartan, H., and S. Eilenberg, *Homological Algebra* (Princeton, 1955).

⁴ Nagata, M., "A General Theory of Algebraic Geometry over Dedekind Rings II," *Am. J. Math.*, **80**, 382-420 (1958).

ON THE FALSITY OF EULER'S CONJECTURE ABOUT THE NON-EXISTENCE OF TWO ORTHOGONAL LATIN SQUARES OF ORDER $4t + 2^*$

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1. *Introduction.*—The purpose of this paper is to prove a general theorem on the existence of pairwise orthogonal Latin squares (p.o.l.s.) of a given order and to give a counter example to Euler's conjecture³ that there do not exist two p.o.l.s. of order $4t + 2$.

2. *Definitions.*—An arrangement of v objects (called treatments) in b sets (called blocks) will be called a pairwise balanced design of index unity and type $(v; k_1, k_2, \dots, k_m)$ if each block contains either k_1, k_2, \dots , or k_m treatments which are all distinct ($k_i \leq v, k_i \neq k_j$), and every pair of distinct treatments occurs exactly in one block of the design. If the number of blocks containing k_i treatments is b_i , then clearly