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Projective resolutions of Weyl modules

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ABSTRACT We present a projective resolution of the two-rowed Weyl module, using techniques of supersymmetric algebra.

Section 1. Introduction

The projective resolution of Weyl modules has been described, in the case of two-rowed modules, in ref. 1, where the technique used leaned heavily on the construction of an arithmetic Koszul complex and on the exactness of certain sequences of skew-shapes. This method proved unwieldy in the attack of the analogous problem for an arbitrary number of rows.

We develop here an altogether different and simple technique which gives the entire characteristic-free finite resolution in one step and which in addition allows an explicit description of a basis of syzygies in each dimension.

We expect the present technique to extend to give a characteristic-free resolution of Weyl modules of an arbitrary number of rows. Because of the timeliness of the method, we present the two-rowed case first. The present paper is an application of the methods of supersymmetric algebra developed in refs. 2 and 3.

Section 2. The Weyl Module

We follow the notation of refs. 2 and 3.

Let A be a linearly ordered proper alphabet. We denote by Super$[A; k]$ the component of degree k of the supersymmetric algebra Super$[A]$. We denote by W both a word in Mon(A) and a monomial in Super$[A]$.

We consider proper linearly ordered signed alphabets L and P of letters and places, respectively, endowed with sufficiently many positively and negatively signed letters and places in arbitrarily prescribed orders to allow for the computations that follow. The linear order in either of the alphabets L and P will be denoted by $\geq$. Throughout this paper, the letters p and q will denote positive integers such that $p \geq q$.

We denote by a and b two places such that $a < b$. The module M_1 is defined as the submodule of the supersymmetric algebra Super$[L|\{a, b\}]$ spanned by all elements of the form $(W|a^{(p)})(W'|b^{(q)})$, where W and W' are monomials in Super$[L]$.

PROPOSITION 1. *For any nonnegative integers* p *and* q *the map*

$$\mathrm{W} \otimes \mathrm{W}' \mapsto (\mathrm{W}|\mathrm{a}^{(\mathrm{p})})(\mathrm{W}'|\mathrm{b}^{(\mathrm{q})})$$

extends to an isomorphism of the module Super[L; p] $\otimes$ *Super*[L; q] *onto the module* M_1. *Thus, monomials of the form*

$$(\mathrm{W}|\mathrm{a}^{(\mathrm{p})})(\mathrm{W}'|\mathrm{b}^{(\mathrm{q})})$$

are a basis of the module M_1, *as* W *and* W′ *range over all monomials of length* p *and* q, *respectively, in Super*[L].

Proof. Obvious.

PROPOSITION 2. *A basis of the module* M_1 *is given by the monomials*

$$(\mathrm{W}''|\mathrm{a}^{(\mathrm{p})}\mathrm{b}^{(\mathrm{r})})(\mathrm{W}'''|\mathrm{b}^{(\mathrm{s})}), \qquad [1]$$

where r *and* s *range over all nonnegative integers such that* r + s = q *and where* W″ *and* W‴ *are any monomials in Super*[L] *such that the diagram* (W″, W‴) *is standard.*

Proof. Immediate from the standard basis theorem of ref. 2.

Let $\{x_1, x_2, x_3, \ldots\}$ be an infinite sequence of negatively signed places such that $x_1 < x_2 < x_3 < \ldots$. We define an operator ∂_1 from M_1 to Super$[L|P]$ by setting

$$\partial_1(W|a^{(p)})(W'|b^{(q)}) = (W|x_1x_2x_3 \ldots x_p)(W'|x_1x_2x_3 \ldots x_q).$$

The image of the operator ∂_1 will be denoted as M_0.

PROPOSITION 3. *The map* ∂_1 *maps standard bitableaux into standard bitableaux.*

Proof. Obvious.

Our first objective will be to determine the kernel of the boundary map ∂_1.

THEOREM 1. *The kernel of the boundary map* ∂_1 *is the submodule of the module* M_1 *freely spanned by all standard bitableaux of the form*

$$(\mathrm{W}''|\mathrm{a}^{(\mathrm{p})}\mathrm{b}^{(\mathrm{r})})(\mathrm{W}'''|\mathrm{b}^{(\mathrm{s})}), \qquad [1]$$

where r + s = q, *where* r > 0, *and where the diagram* (W″, W‴) *is standard in Super*[L]. *The module* M_1 *is also spanned by the bitableaux of the form*

$$(\mathrm{W}''|\mathrm{a}^{(\mathrm{p})}\mathrm{b}^{(\mathrm{r})})(\mathrm{W}'''|\mathrm{b}^{(\mathrm{s})}),$$

where W″ *and* W‴ *are arbitrary words in Super*[L].

Proof.

Case 1. $r = 0$. In this case, the image of bitableau **1** under the boundary map is the standard bitableau

$$(W|x_1x_2x_3 \ldots x_p)(W''|x_1x_2x_3 \ldots x_q),$$

and we know that standard bitableaux are nonzero (2).

Case 2. $r > 0$. In this case, the image of bitableau **1** under the boundary map is computed as follows. Let

$$u = x_1x_2x_3 \ldots x_q,$$

and let

$$\Delta u = \sum u_{(1)} \otimes u_{(2)}. \qquad [2]$$

We write

$$\sum{}' u_{(1)} \otimes u_{(2)}$$

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to indicate the sum of those terms on the right side of Eq. **2** (and only those) where $u_{(1)}$ has degree r and $u_{(2)}$ has degree s. In this notation, the image of bitableau **1** under the boundary map ∂_1 is

$$\sum{}' \pm (W''|x_1x_2x_3 \dots x_p u_{(1)})(W'''|u_{(2)}),$$

for a suitable choice of signs.

Since $r > 0$, every word $x_1x_2x_3 \dots x_p u_{(1)}$ in this sum contains two repeated negatively signed letters, and therefore it vanishes.

Every element of the kernel of ∂_1 is a linear combination of elements of the form **1**. We may assume that only terms with $r = 0$ occur in such a linear combination, otherwise there is nothing to prove. But then, the image of a linear combination of such terms is mapped into a linear combination of standard tableaux (by *Case 1*), and, again by the standard basis theorem, such a linear combination vanishes if and only if the coefficient of each of the terms vanishes. This concludes the proof of the first assertion.

To prove the second part of the statement, it suffices to invoke again the standard basis theorem. Any tableau of the form

$$(W''|a^{(p)}b^{(r)})(W'''|b^{(s)})$$

is a linear combination with integer coefficients of standard tableaux of the form

$$(W''''|a^{(p)}b^{(t)})(W'''''|b^{(s)})$$

with $t \geq r$, hence the conclusion.

Section 3. The Resolution

We next define the modules M_{k+1} for $1 \leq k \leq q$.

Choose a new positive alphabet R, having an infinite set of letters $u_1, u_2, \dots$ such that $u_1 < u_2 < \dots$, and consider the underlying module of the algebra Super[R] as a module over Super[$L|\{a, b\}$].

A **Z**-basis of the Super[$L|\{a, b\}$]-module Super[R] is given by all elements of the form

$$(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)} \dots u_n^{(\gamma)},$$

where $\alpha, \beta, \dots, \gamma$ are nonnegative integers and where the bitableau $(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})$ is standard, as in *Proposition 2*.

The module M_{k+1} is freely spanned by all elements of Super[R] of the form

$$(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_3^{(\gamma)} \dots u_{k-1}^{(\varphi)}u_k^{(\varpi)},$$

where the bitableau $(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})$ is standard, where the integers $\alpha, \beta, \gamma, \dots, \varphi, \varpi$ are positive, and where $\pi = p + \alpha + \beta + \dots + \varphi + \varpi$, and $\pi + \rho + \sigma = p + q$.

The boundary operator ∂_{k+1} is defined as follows:

$$\begin{aligned}
&\partial_{k+1}(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_3^{(\gamma)} \dots u_{k-1}^{(\varphi)}u_k^{(\varpi)} \\
&= (W''|a^{(\pi-\alpha)}b^{(\alpha)}b^{(\rho)}) \\
&\times (W'''|b^{(\sigma)})u_1^{(\beta)}u_2^{(\gamma)} \dots u_{k-2}^{(\varphi)}u_{k-1}^{(\varpi)} \\
&- (W''|a^{(\pi)}b^{(\rho)}) \\
&\times (W'''|b^{(\sigma)})u_1^{(\alpha)}u_1^{(\beta)}u_2^{(\gamma)} \dots u_{k-2}^{(\varphi)}u_{k-1}^{(\varpi)} \\
&+ (W''|a^{(\pi)}b^{(\rho)}) \\
&\times (W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_2^{(\gamma)} \dots u_{k-2}^{(\varphi)}u_{k-1}^{(\varpi)} - \dots \\
&+ \dots \\
&\pm (W''|a^{(\pi)}b^{(\rho)}) \\
&\times (W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_3^{(\gamma)} \dots u_{k-1}^{(\varphi)}u_{k-1}^{(\varpi)}.
\end{aligned}$$

With this definition of the boundary, we have

$$\partial_k\partial_{k+1}(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_3^{(\gamma)} \dots u_{k-1}^{(\varphi)}u_k^{(\varpi)} = 0.$$

We shall now prove that the complex defined by M_k and ∂_k is exact; in fact, we shall find a basis of syzygies for each dimension. To this end, we define a homotopy operator S_k, mapping M_k to M_{k+1}, as follows. For $k > 0$ set

$$\begin{aligned}
&S_k(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_3^{(\gamma)} \dots u_{k-1}^{(\varphi)} \\
&= (W''|a^{(\pi+\rho)})(W'''|b^{(\sigma)})u_1^{(\rho)}u_2^{(\alpha)}u_3^{(\beta)} \dots u_k^{(\varphi)}
\end{aligned}$$

whenever ρ is a positive integer and 0 otherwise. The operator S_0 maps

$$(W|x_1x_2x_3 \dots x_p)(W'|x_1x_2x_3 \dots x_q)$$

to

$$(W|a^{(p)})(W'|b^{(q)}).$$

Again, a simple computation shows that

$$\partial_{k+2}S_{k+1} + S_k\partial_{k+1} = \text{identity}.$$

We thus obtain our main result:

THEOREM 2. *The complex defined by* $\mathrm{M_k}$ *and* ∂_k *is exact.*

Section 4. Bases of Syzygies

We shall now display an explicit resolution for the Weyl module M_0 and a basis for its syzygies in each dimension. Recall that p and q are positive integers such that $p \geq q$. We have

THEOREM 3. *The complex* $\{\mathrm{M_k}, \partial_\mathrm{k}\}_{\mathrm{k}\geq 1}$ *is a projective resolution of the Weyl module* $\mathrm{M_0}$ *over the Schur algebra of degree* p + q, *with the map* $\partial_1:\mathrm{M_1} \to \mathrm{M_0}$ *as augmentation. The* (k + 1)-*dimensional syzygy module of* $\mathrm{M_0}$ *is the submodule of* $\mathrm{M_{k+1}}$ *freely spanned by the elements*

$$(\mathrm{W''}|\mathrm{a}^{(\pi)})(\mathrm{W'''}|\mathrm{b}^{(\sigma)})\mathrm{u}_1^{(\alpha)}\mathrm{u}_2^{(\beta)}\mathrm{u}_3^{(\gamma)} \dots \mathrm{u}_{k-1}^{(\varphi)}\mathrm{u}_k^{(\varpi)},$$

where the diagram (W″, W‴) *is standard.*

Proof. Since each module M_k, for positive k, is a direct sum of tensor products of divided powers, we know by ref. 1 that these modules are projective over the Schur algebra of degree $p + q$. The exactness of the complex

$$\dots \to M_{k+1} \to M_k \dots \to M_2 \to M_1 \to M_0 \to 0$$

asserts that $\{M_k, \partial_k\}_{k\geq 1}$ is indeed a resolution. As for the statement about syzygies, this follows from the facts that $\{S_k\}$ is a splitting homotopy and that the nonzero elements of the image under S_k of the basis elements for M_k are linearly independent.

In the language of ref. 2, this amounts to saying that the resolution is invariant under polarization of the letters L.

Section 5. Extension to Skew Shapes

The discussion is similar to the preceding, except that the module M_0 is now defined as follows. Choose a set of t negative letters $y_1, y_2, \dots, y_t$ such that $y_1 < y_2 < \dots < y_t$ and such that all other letters of L follow y_t. The module M_0

is freely spanned by all standard bitableaux of the form $(y_1y_2 \ldots y_tW|x_1x_2x_3 \ldots x_{p+t})(W'|x_1x_2x_3 \ldots x_q)$.

The only change required in the proof is in the definition of the modules M_{k+1} for $k \geq 1$, as well as in the definition of the homotopy operator. The modules M_{k+1} are freely spanned by elements of the form

$$(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)})u_1^{(\alpha)}u_2^{(\beta)}u_3^{(\gamma)} \ldots u_{k-1}^{(\varphi)}u_k^{(\varpi)},$$

where $\varpi > t$, where $\alpha, \beta, \ldots, \varphi$ are positive, and where as above we have $\pi = p + \alpha + \beta + \ldots + \varphi + \varpi$, and $\pi + \rho + \sigma = p + q$.

All the homotopy operators S_k remain unchanged for $k > 1$. The operator S_1 is redefined as follows:

$$S_1(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)}) = (W''|a^{(\pi+\rho)})(W'''|b^{(\sigma)})u_1^{(\rho)}$$

whenever $\rho > t$, and

$$S_1(W''|a^{(\pi)}b^{(\rho)})(W'''|b^{(\sigma)}) = 0$$

otherwise.

The operator S_0 is defined on all tableaux of the form $(y_1y_2 \ldots y_tW|x_1x_2x_3 \ldots x_{p+t})(W'|x_1x_2x_3 \ldots x_q)$, where the Young diagram $(y_1y_2 \ldots y_tW, W')$ is standard. Say that $W = c_1c_2 \ldots c_p$ and $W' = d_1d_2 \ldots d_q$. By the definition of a standard (super) Young diagram, there exists a smallest index r such that $d_r \leq c_1 < d_{r+1}$. Consider the Young diagrams $D = (d_1d_2 \ldots d_rc_1c_2 \ldots c_p, d_{r+1}d_{r+2} \ldots d_q)$ and $E = (a^pb^r, b^{q-r})$. We define

$$S_0(y_1y_2 \ldots y_tW|x_1x_2x_3 \ldots x_{p+t})(W'|x_1x_2x_3 \ldots x_q) = \mathrm{Tab}(D|E).$$

With these changes, the statements of *Theorems 2* and *3* remain valid for skew tableaux.

Section 6. Applications to the Bracket Algebra

The preceding results yield a projective resolution of the submodule of the bracket algebra (3) spanned by all products of two brackets. One simply interprets a bracket of length n as $[W] = (W|x_1x_2 \ldots x_n)$, setting $p = q = n$ above. In this way, we obtain a resolution of the degree two component of the ring of coordinates of the Grassmannian.

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