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D. A. Buchbaum, Gian-Carlo Rota

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A new construction in homological algebra

(representation theory/invariant theory/supersymmetric algebras/resolutions)

D. A. Buchsbaum* AND Gian-Carlo Rota†

*Brandeis University, Waltham, MA 02254; and †Massachusetts Institute of Technology, Cambridge, MA 02139

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ABSTRACT We present a generalization of the classical bar construction with applications to resolutions of Weyl modules.

Section 1. Introduction

One of the fundamental constructions of homological algebra is the bar resolution, which has been successfully applied to obtain projective resolutions in several contexts (e.g., see ref. 1). In a previous announcement (2), we applied the bar resolution to obtain a characteristic-free projective resolution of two-rowed Weyl modules.

In trying to extend the method to obtain a characteristic-free projective resolution of the general Weyl module, we were led to introduce a construction which amounts to a very wide generalization of the bar construction. We believe this construction, which we propose to call the differential bar complex, to be of independent interest and to be potentially applicable to a large variety of contexts.

The differential bar complex is defined in Section 2. The main components of this construction are the exterior algebra, Ext(S), on a set of free generators, S, called the separators; an algebra, A; an A module, M; and the free product of Ext(S) with A. We give a few straightforward examples and refinements of this main idea.

In Section 3, we present the resolution of the two-rowed Weyl module done in ref. 2, using the language of Section 2.

In Section 4, we prove that a bar complex with two separators can be used to obtain a characteristic-free projective resolution of a restricted class of three-rowed Weyl modules. We believe that a characteristic-free projective resolution of all Weyl modules can be obtained by systematic use of the differential bar complex.

Section 2. The Differential Bar Complex

Let A be an associative algebra with identity, and let S be a set whose elements will be called separators. We denote by Ext(S) the free exterior algebra on the separators. The algebra Ext(S) has a natural Z2 grading: if m is a monomial in Ext(S), that is, a product of generators, we set |m| = 0 if m is the product of an even number of generators, and |m| = 1 if m is the product of an odd number of generators.

The free product of the algebra A and the algebra Ext(S) will be called the bar algebra on the algebra A, with set of separators S, and denoted by Bar(A; S).

The algebra Bar(A; S) inherits a Z2 grading, defined as follows. Every element of Bar(A; S) is a linear combination of elements of the form

\[ W = w_1m_1w_2m_2 \ldots w_km_k, \]

where the \( m_i \) are nonzero monomials in Ext(S), and where the \( w_i \) are elements of A. We set |W| = 0 if |m_1m_2 \ldots m_k| = 0 and |W| = 1 if |m_1m_2 \ldots m_k| = 1. One extends this definition by linearity to a Z2 grading of the algebra Bar(A; S).

For every finite subset T of the set S of separators, the underlying module of the algebra Bar(A; S) has a grading, which will be called the T grading of Bar(A; S) and which is defined as follows. The submodule Bar(A; S; T, i) of degree i is spanned by all elements of the form in Eq. 1, where the integer i equals the total number of occurrences of separators in the set T in the sequence (m_1, m_2, \ldots, m_k). Clearly, the submodule Bar(A; S; T, i) is a left A module.

Recall that for every separator x, there exists a unique antiderivation \( \partial_x \) of the algebra Ext(S), such that \( \partial_x(x) = 1 \) (where 1 is the identity of the exterior algebra Ext(S)), and \( \partial_y(y) = 0 \) for \( y \neq x \) in S and \( y \neq x \). Recall also that \( (\partial_x)^2 = 0 \) and \( \partial_x\partial_y = -\partial_y\partial_x \).

It can be shown that the antiderivation \( \partial_x \) uniquely extends to an antiderivation of the Z2-graded algebra Bar(A; S), again denoted by \( \partial_x \), as defined above. If W is as in Eq. 1, set \( \partial_x(W) = w_1\partial_x(m_1)w_2m_2 \ldots w_km_k + \ldots + (-1)^{|m_1|+\ldots+|m_{k-1}|}w_1m_1\partial_x(m_2)w_3m_3 \ldots w_km_k \).

The antiderivation \( \partial_x \) is well-defined. Again, we have \( (\partial_x)^2 = 0 \) and \( \partial_x\partial_y = -\partial_y\partial_x \). If T is a nonempty finite subset of S, the operator \( \partial_T = \sum_x \partial_x \) as \( x \) ranges over T, is called the T-boundary operator. The boundary operator \( \partial_T \) maps Bar(A; S; T, i + 1) into Bar(A; S; T, i), for \( i = 0, 1, 2, \ldots \).

Now let M be a left A module. If w is an element of A, we denote the action of w on M by w(A). The free bar module of the A module M, with set of separators S, is defined as follows. It consists of a left A module Bar(M, A; S) and of an antiderivation, again denoted \( \partial_x \), of the module Bar(M, A; S) into itself, defined for each separator x. These are defined as follows.

The module Bar(M, A; S) is spanned by elements of the form \( W \otimes v \), where W is as in Eq. 1 and v is in M. In addition, the following relations are assumed to be satisfied in Bar(M, A; S). If \( m_k = 1 \), then

\[ w_1m_1w_2m_2 \ldots w_km_k \otimes v = w_1m_1w_2m_2 \ldots m_{k-1} \otimes w_k(v). \]  \[ [2] \]

Next, the antiderivation \( \partial_x \) is defined by setting

\[ \partial_x(w_1m_1w_2m_2 \ldots w_km_k \otimes v) = \partial_x(w_1m_1w_2m_2 \ldots w_km_k) \otimes v. \]

The following special case deserves notice. If \( m_k = x \), then
\( \partial_\delta(w_1m_1w_2m_2 \ldots w_km_k) \otimes v \)

\( = \partial_\delta(m_1w_1m_2w_2 \ldots m_{k-1}w_k)x \otimes v \)

\( + (-1)^{|m_1|+|m_2|+\ldots+|m_{k-1}|}m_1w_1m_2w_2 \ldots m_{k-1}w_k\partial_\delta(x) \otimes v \)

\( = \partial_\delta(m_1w_1m_2w_2 \ldots m_{k-1}w_k)x \otimes v \)

\( + (-1)^{|m_1|+|m_2|+\ldots+|m_{k-1}|}m_1w_1m_2w_2 \ldots m_{k-1}w_k\partial_\delta(v) \).

Now let \( M' \) be a submodule of \( \text{Bar}(M, A; S) \) such that \( \partial_\delta(M') \) is a submodule of \( M' \), for every separator \( x \). Then \( \partial_x \) is defined on the quotient module \( \text{Bar}(M, A; S)/M' = M'' \). Any submodule \( M'' \) of \( M'' \) which is closed under \( \partial_x \), for every separator \( x \) will be called a bar module associated with \( \text{Bar}(M, A; S; T, i) \).

**Theorem 1.** Let \( M'' \) be a bar module associated with \( \text{Bar}(M, A; S; T, i) \). Then the sequence

\[ \cdots \to M_2 \to M_1 \to M_0 \to 0 \]  

is a complex, where each map is a restriction of \( \partial_T \). (b) Let \( T' \) be a nonempty finite subset of \( S \), disjoint from \( T \), and let \( M_0 \) be the submodule of \( M'' \cap \text{Bar}(M, A; S) \) spanned by all elements of the form \( W \otimes v \), with \( W \) belonging to \( \text{Bar}(A; S; T, i) \). Then the sequence

\[ \cdots \to M_2 \to M_1 \to M_0 \to 0, \]  

where each map is a restriction of \( \partial_T \), is a complex of complexes.

We will call the complex \( 3 \) the bar complex associated with the bar module \( M'' \). The notions of bar module and bar complex are closely related and will be used interchangeably.

**Example 1.** Let \( S \) be a one-element set, containing the element \( x \). Then the module \( \text{Bar}(M, A; S) \) is spanned by all elements of the form

\[ w_1w_2xw_3 \ldots w_{l}x \otimes v \]  

and the derivation \( \partial_x \) is computed as follows:

\( \partial_x(w_1w_2xw_3 \ldots w_{l}x \otimes v) \)

\( = (w_1w_2xw_3 \ldots w_{l}x \otimes v) - (w_1w_2w_3xw_4 \ldots w_{l}x \otimes v) \)

\( + (w_1w_2w_3w_4xw_5 \ldots w_{l}x \otimes v) - \ldots \)

\( + \ldots (-1)^{l-1}w_1xw_3 \ldots w_{l-1}x \otimes v. \)

Thus, the free bar module on \( M \) with a single separator gives rise to the classical bar resolution, as one sees by replacing the symbol \( x \) by the symbol \( | | \).

**Example 2.** Let \( A \) be a connected graded algebra with identity. The submodule of \( 5 \) spanned by elements of the form \( \delta_\partial \), where \( w_1, w_2, \ldots, w_k \) are all of positive degree, gives, by Theorem 1, the classical normalized bar construction of \( A \) and \( M \) with set of separators \( S \).

**Example 3.** We consider next the simplest example where two separators are used. Let \( M \) be a module over an algebra \( A \), and let \( S = \{x, y\} \) be a set of two separators. The typical elements of the submodules \( \text{Bar}(M, A; S, i) \) for \( i = 0, 1, 2, 3 \), are as follows.

For \( i = 0 \) we obtain the elements of \( M \).

The elements of \( \text{Bar}(M, A, S, 1) \) are of one of the forms \( axm \) and \( bym \), where \( a \) and \( b \) are in \( A \) and \( m \) is in \( M \). The boundary map \( \partial_\delta = \partial_x + \partial_\delta \) maps \( axm \) to \( am \) and \( bym \) to \( bm \).

The elements of \( \text{Bar}(M, A, S, 2) \) are of one of the forms \( ax^2xm \), \( ax^2ym \), \( ay^2xm \), and \( ay^2ym \), and we have

\[ \partial_\delta(abxm) = abxm - axbm \]

\[ \partial_\delta(abym) = abym \]

\[ \partial_\delta(ayxbm) = ayxbm - axbm \]

\[ \partial_\delta(ayxym) = abym - aybm \]

\[ \partial_\delta(abxm) = abxm - axbm \]

\[ \partial_\delta(abym) = abym - aybm \]

\[ \partial_\delta(ayxbm) = axbm - aybm \]

\[ \partial_\delta(ayxym) = abxm - aybm \]

\[ \partial_\delta(abxm) = abxm - axbm \]

\[ \partial_\delta(abym) = abym - aybm \]

**Example 4.** With \( A \) and \( M \) as above, assume that the degree of \( w_1 \) is \( \geq t \). One then obtains the \( t^* \) submodule (and hence, also the \( t^* \) complex) of the normalized bar construction. If in addition \( M \) is a graded module, if a positive integer \( n \) is fixed, and if in expression \( S \) it is assumed that \( \deg(w_1) + \deg(w_2) + \ldots + \deg(w_l) + \deg(v) = n \), one obtains a bar module which will be called the \( t^* \)-graded subbar of degree \( n \).

**Section 3. Resolution of the Two-Rowed Weyl Module**

We restate in the present language the resolution of the Weyl module associated to a two-rowed skew shape obtained in ref. 2, section 5. Recall that the Weyl module itself was denoted \( M_i \), and that the other modules of the resolution were denoted \( M_i \) for \( i = 1, 2, \ldots \). Let \( A \) be a linearly ordered proper alphabet. We denote by \( W \) both a word in \( \text{Mon}(A) \) and a monomial in \( \text{Super}[A] \). We consider proper linearly ordered signed alphabets \( L \) and \( P \) of letters and places, respectively, endowed with sufficiently many positively and negatively signed letters and places in arbitrarily prescribed orders to allow for the computations that follow. The linear order in \( L \) and \( P \) will be denoted by \( \leq \). We denote by \( a \) and \( b \) two places such that \( a < b \). The module \( M_1 \) is defined as the submodule of the supersymmetric algebra \( \text{Super}[L][a, b] \) spanned by all elements of the form

\[ (W[a\partial], W[b\partial]), \]

where \( W \) and \( W' \) are monomials in \( \text{Super}[L] \).

Recall that for any nonnegative integers \( p \) and \( q \), the map

\[ W \otimes W' \mapsto (W[a\partial], W[b\partial]), \]

extends to an isomorphism of the module \( \text{Super}[L; p] \otimes \text{Super}[L; q] \) onto the module \( M_1 \). Thus, monomials of the form

\[ (W[a\partial], W[b\partial]), \]

are a basis of the module \( M_1 \), where \( W \) and \( W' \) range over all monomials of lengths \( p \) and \( q \), respectively, in \( \text{Super}[L] \).
Recall that another basis of the module $M_1$ is given by the monomials
\[ (W^m|a^p|b^q)(W^m|b^q), \]
where $r$ and $s$ range over all nonnegative integers such that $s \leq p$ and $r + s = q$ and where $W^m$ and $W^n$ are any monomials in $\text{Super}[L]$ such that the diagram $(W^m, W^n)$ is standard. The module $M_2$ is defined as follows. Choose an infinite sequence of negatively signed places $\{x_1, x_2, x_3, \ldots\}$ such that $x_1 < x_2 < x_3 < \ldots$, and a set of $t$ negative letters $y_1, y_2, \ldots, y_t$, such that $y_1 < y_2 < \ldots < y_t$ and such that all other letters of $L$ follow $y$. The module $M_0$ is freely spanned by all double standard tableaux of the form
\[ (y_1y_2 \ldots y_tW|x_1x_2x_3 \ldots x_{p+r})(W^m|x_1x_2x_3 \ldots x_q), \]
where the Young diagram $(y_1y_2 \ldots y_tW, W')$ is standard.

A boundary operator $\partial_1$ from $M_1$ to $\text{Super}[L|P]$ was defined by setting
\[ \partial_1(W|a^p)(W'|b^q) = (y_1y_2 \ldots y_tW|x_1x_2x_3 \ldots x_{p+r})(W'|x_1x_2x_3 \ldots x_q). \]
The image of the operator $\partial_1$ was denoted as $M_0$.

The kernel of the boundary map $\partial_1$ is the submodule of the module $M_1$ freely spanned by all standard bitableaux of the form
\[ (W^m|a^p|b^q)(W^m|b^q), \]
where $r + s = q$, where $r > t$.

We next define the modules $M_{k-1}$ for $k \geq 2$. To this end, we consider the divided power algebra $\text{Div}(Z)$ generated by a single variable $Z$. The algebra $\text{Div}(Z)$ acts on the module $\text{Super}[L|[a,b]]$ by as follows:
\[ Z^{(a)}((W|a^p)(W'|b^q)) = D^{(a)}(W|a^p)(W'|b^q), \]
where $D^{(a)}$ is the $a$th divided power of the polarization operator $D_{ba}$.

The modules $M_{k+1}$ are submodules of the free bar module $\text{Super}[L|[a,b]]$ over the algebra $\text{Div}(Z)$, with a single separator $x$, spanned by elements of the form
\[ Z^{(a)}xZ^{(b)}x \ldots xZ^{(a)}x(W^m|a^p|b^q)(W^m|b^q), \]
where the separator $x$ appears $k$ times, and where $\alpha, \beta, \ldots, \gamma$ are positive integers, and where the tableau $(W^m|a^p|b^q)$ ($W^m|b^q$) is standard, as in ref. 2.

The module $M_{k+1}$ is freely spanned by all elements of the form
\[ Z^{(a)}xZ^{(b)}xZ^{(c)}x \ldots xZ^{(a)}xZ^{(b)}x(W^m|a^p|b^q)(W^m|b^q), \]
where $\alpha > t$, where $\beta, \gamma, \ldots, \omega$ are positive, and where as above we have $p = \alpha + \beta + \ldots + \varphi + \omega$, and $p + \sigma = p + q$.

All boundary operators $\partial_{k+1}$ are defined as $\partial_1$. The resulting complex is exact, as we showed in ref. 2. More succinctly, what we have described is the following. In $\text{Super}[L|[a,b]]$, the submodule, $C$, of total content $p + q$ in $[a,b]$ is stable under the action of $\text{Div}(Z)$ and is graded by the content in the place $b$. The above complex is the $(t + 1)^{st}$-graded strand of degree $q$ of $\text{Bar}(C, \text{Div}(Z); x)$.

Section 4. Application to Three-Rowed Weyl Modules

The resolution of the Weyl module associated with the three-rowed skew shape always has a filtration, whose bottom layer can be described by a multiple bar construction.

The Weyl module $M_{-1}(p, q; r; t_1, t_2)$ associated with the three-rowed skew shape $(p + t_1 + t_2, q + t_1 + t_2, r)/(t_1 + t_2, t_2, 0)$, namely, the shape represented by the Ferrers matrix
\[
\begin{array}{cccccccc}
0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\end{array}
\]

or the diagram
\[
\begin{array}{cccccccc}
t_2 & t_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]
is defined as follows. Choose an infinite sequence of negatively signed places $\{x_1, x_2, \ldots\}$ such that $x_1 < x_2 < \ldots$ and a set of $t_1 + t_2$ negative letters $y_1, y_2, \ldots, y_{t_1+t_2}$ such that $y_1 < y_2 < \ldots < y_{t_1+t_2}$ and such that all other letters of $L$ follow $y$. The module $M_{-1}(p, q; r; t_1, t_2)$ is freely spanned by all double standard tableaux of the form
\[ (y_1y_2 \ldots y_{t_1+t_2}W|x_1x_2x_3 \ldots x_{p+r})(W^m|x_1x_2x_3 \ldots x_q), \]
where $W, W'$, and $W^m$ are words in $\text{Super}(L)$, and the Young diagram on the left side is standard.

We now choose three positive places $a < b < c$. Consider the algebra $\text{Div}(D_{ba}, D_{bc}, D_{ca})$ generated by the polarization operators $D_{ba}, D_{bc}, D_{ca}$ and their divided powers. In this algebra we have the identities
\[ D_{ca} = D_{cb}D_{ba} - D_{ba}D_{cb} \]
and
\[ D^{(a)}_{cb}D^{(b)}_{ba} = \sum_{k=0}^{a} D^{(b-a+k)}_{ba}D^{(a-k)}_{cb}D^{(c)}_{cb}. \]

We consider now the free bar module $\text{Bar}(\text{Super}(L|[a,b,c]))$, $A(Z_{ba}, Z_{bc}, Z_{ca}; (x,y))$, where $x$ and $y$ are the two separators; the algebra $A(Z_{ba}, Z_{bc}, Z_{ca})$ is the associative noncommutative algebra generated by the variables $Z_{ba}, Z_{bc},$ and $Z_{ca}$ with divided powers, subject to the commutation relations $Z_{ba}Z_{cb} = Z_{cb}Z_{ca}$ and $Z_{ca}Z_{ba} = Z_{ba}Z_{ca}$; and the algebra $A(Z_{ba}, Z_{bc}, Z_{ca})$ acts on the module $\text{Super}(L|[a,b,c])$ by letting $Z_{ba}$, $Z_{ca}$, and $Z_{cb}$ be the polarization operators $D_{ba}$, $D_{ca}$, and $D_{cb}$, respectively.

We define a submodule quotient of the bar module $\text{Bar}(\text{Super}(L|[a,b,c]))$, $A(Z_{ba}, Z_{bc}, Z_{ca}; (x,y))$ as follows. We impose the relations
\[ Z^{(a)}_{cb}x = \sum_{k=0}^{a} Z^{(b-a+k)}_{ba}xZ^{(a-k)}_{cb}Z^{(c)}_{ca}, \]
and
\[ Z_{cb}x = xZ_{cb}. \]

The module $M_0(p, q; r; t_1, t_2)$ is freely spanned by all elements of the form
Note that $F_{r-k}(M) = M(p, q, r; t_1, t_2)$. Stage 4. Observe that $F_0(M) = M(p, q, r; t_1, r - k)$. That is, $F_0(M)$ is the complex associated with the skew shape whose Ferrers diagram is

\[
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

(S_\lambda).

Stage 5. The complexes $F_k(M)/F_{k-1}(M)$ for $k \geq 1$ are isomorphic to the complexes $M(p, q + r - k + 1, k - 1; t_1 + r - k + 1)$, namely, the complexes associated with the skew shape whose Ferrers diagram is

\[
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

(\Gamma_{k-1}).

The isomorphism between $F_0(M)/F_{k-1}(M)$ and $M(p, q + r - k + 1, k - 1; t_1 + r - k + 1)$ results from the observation that every term of $F_0(M)/F_{k-1}(M)$ is of the form $Z^m G^m_{\lambda + \mu}$, where $m$ is an element of $M(p, q + r - k + 1, k - 1; t_1 + r - k + 1)$. Thus the isomorphism is defined by mapping $Z^m G^m_{\lambda + \mu}$ to $m$.

It is essential to note that this isomorphism involves a dimension shift of 1, namely, the terms of $F_0(M)/F_{k-1}(M)$ of dimension $i$ are mapped to terms of dimension $i - 1$ in $M(p, q + r - k + 1, k - 1; t_1 + r - k + 1)$.

Step 6. We next prove that the complexes $F_k(M)$ are resolutions of the modules $M_{-1}(p, q, r; t_1, r - k)$ and that $F_k(M)/F_{k-1}(M)$ are resolutions of $M_{-1}(p, q + r - k + 1, k - 1; t_1 + r - k + 1)$. This is done by induction on $r - t_2$, starting with $r - t_2 = 0$, and repeated use of the exact sequences:

\[
0 \rightarrow \Gamma_{k-1} \rightarrow \Sigma_{k-1} \rightarrow \Sigma_k \rightarrow 0
\]

(\text{E}).

(The proof of the exactness of E is to be found in ref. 3.)

It is clear that $F_0(M)$ and $F_1(M)/F_0(M)$ are resolutions. The long exact homology sequence derived from the short exact sequence,

\[
0 \rightarrow F_0(M) \rightarrow F_1(M) \rightarrow F_1(M)/F_0(M) \rightarrow 0,
\]

gives us immediately the vanishing of the homology of $F_1(M)$ in dimensions greater than or equal to 2. In addition, $H_1(F_1(M))$ is the kernel of the map of $H_1(F_0(M)/F_0(M))$ into $H_0(D_0(M))$, and $H_0(F_0(M))$ is the cokernel of that map. Using the exactness of E for $k = 1$, we see that this map has zero kernel, and its cokernel is $M(p, q, r; t_1, r - 1)$. This establishes that $F_1(M)$ is a resolution but also establishes that $F_0(M)/F_1(M)$ is a resolution, since the $r - t_2$ value for this case is equal to 1. We may therefore proceed to the next exact sequence,

\[
0 \rightarrow F_1(M) \rightarrow F_2(M) \rightarrow F_2(M)/F_1(M) \rightarrow 0,
\]

and proceed as before. Since $F_{r-k}(M) = M(p, q, r; t_1, t_2)$, we have the proof of our theorem.

The argument above can be streamlined by resorting to the spectral sequence associated with the filtration, but the
essential ingredients would still be the induction hypothesis and the exact sequences (E).

In Theorem 2, the condition that $r \leq t_1 + t_2 + 1$ is essential, because we do not have the exact sequence (E) if $r > t_1 + t_2 + 1$. However, we hope to show in a later communication that the resolution of the Weyl module has a filtration of which the complex $M(p, q, r; t_1, t_2)$ is the bottom layer.

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