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On Lifting Maps between Weyl Modules: Can Bad Shapes Be Resolved by Better Shapes?

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1. INTRODUCTION

Ever since the appearance of [A-B.2], in which the existence of finite projective resolutions of Weyl modules was proven, there has been a good deal of interest in describing such resolutions explicitly. An explicit resolution of two-rowed skew shapes was given in [A-B.1], but as yet no characteristic-free resolution of the general n-rowed skew shape (or even partition) has been made explicit. (A recent reworking of the two-rowed case by Buchsbaum and Rota [B-R] does seem to offer a hint as to the form of the general resolution, but it is still too early to make any comprehensive positive claims.)

At one point in these investigations, it was observed that three-rowed "bad" shapes have a "resolution" in terms of "better" three-rowed skew shapes and that these resolutions very closely resembled in form the projective resolutions of two-rowed shapes. This observation initiated a program to resolve bad shapes in terms of better shapes. (Definitions of these terms are given in Section 2.) A positive spurt to this campaign was provided by the successful resolution of the worst bad shapes in terms of better ones. The way to proceed to the next step seemed clear. Unfortunately, despite persistent efforts to implement the next step, there always seemed to be some trouble. In characteristic zero, the use of the equivalence of the functors, $\operatorname{Hom}_G(L_\lambda, L_\mu \otimes X) \approx \operatorname{Hom}_G(L_{\lambda/\mu}, X)$, helped to circumvent the

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problems, but it was soon clear that this equivalence is not true in every characteristic. (Here, too, is an interesting problem: we know that the functor $\mathrm{Hom}_G(L_\lambda, L_\mu \otimes X)$ is representable. What is it represented by?) But basically, as we show in Section 3, the obstacle lay in trying to lift a map between bad shapes to a map between associated good shapes. Since this lifting could be effected in characteristic zero, we had to find an arithmetic way to study the liftings of maps to see what was really occurring. Using the method of computing weight modules described in [A-B.2] in connection with calculation of intertwining numbers, we produced an example to show that in general the maps we were trying to lift could not be lifted. In the final section of this paper, we give this example and show that our lifting problem is not solvable in characteristic two.

Because much of the background of this problem has not appeared in print elsewhere, we are including enough of this material to make the solution of the problem understandable. The basic definitions of the Schur and Weyl modules, and the Schur algebras, are those defined and used in [A-B-W, A-B.1, and A-B.2].

2. SOME BACKGROUND MATERIAL

We assume that the reader is familiar with the definitions of Weyl and Schur modules corresponding to a shape matrix A [A-B-W, A-B.1], denoted by K_A and L_A, respectively. (A free module, F, over a commutative ring is tacitly assumed to be given. Hence we generally avoid writing $K_A(F)$, $L_A(F)$, etc.) When A is the shape matrix of a partition λ, we write K_λ (or L_λ), and when A is the matrix of a skew shape λ/μ, we write $K_{\lambda/\mu}$ (or $L_{\lambda/\mu}$). By proving that the modules $K_{\lambda/\mu}$ and $L_{\lambda/\mu}$ have bases consisting of standard tableaux, their universal freeness is established. (This means in particular that if R is a commutative ring and F is a free R-module, we may write $F = R \otimes_{\mathbb{Z}} F_0$, where F_0 is a free abelian group, and we have $K_{\lambda/\mu}(F) = R \otimes K_{\lambda/\mu}(F_0)(L_{\lambda/\mu}(F) = R \otimes L_{\lambda/\mu}(F_0))$. We concentrate on Weyl modules in this paper; everything that we show for these modules can be carried over mutatis mutandis to Schur modules.

Let $\lambda = (\lambda_1, ..., \lambda_n)$, $\mu = (\mu_1, ..., \mu_n)$ be partitions with $\mu \subseteq \lambda$ (i.e., $\mu_i \leqslant \lambda_i$ for $i = 1, ..., n$). Then λ/μ is a skew shape and the Weyl module, $K_{\lambda/\mu}$, is the cokernel of the map

$$\square : \sum_{\nu=1}^{n-1} \sum_{l \geqslant t_\nu} D_{p_1} \otimes \cdots \otimes D_{p_\nu + l} \otimes D_{p_{\nu+1} - l} \otimes \cdots \otimes D_{p_n} \to D_{p_1} \otimes \cdots \otimes D_{p_n},$$

where

$$p_\nu=\lambda_\nu-\mu_\nu \qquad \nu=1,\ldots,n;$$

$$t_\nu=\mu_\nu-\mu_{\nu+1}+1 \qquad \nu=1,\ldots,n-1;$$

$D_p=$ the pth divided power;

and the map

$$D_{p_1}\otimes\cdots\otimes D_{p_\nu+l}\otimes D_{p_{\nu+1}-l}\otimes\cdots\otimes D_{p_n}\to D_{p_1}\otimes\cdots\otimes D_{p_n}$$

is the composition

$$\begin{aligned} &D_{p_1}\otimes\cdots\otimes D_{p_\nu+l}\otimes D_{p_{\nu+1}-l}\otimes\cdots\otimes D_{p_n}\\ &\xrightarrow{1\otimes\cdots\otimes\Delta\otimes\cdots\otimes 1} D_{p_1}\otimes\cdots\otimes D_{p_\nu}\otimes D_l\otimes D_{p_{\nu+1}-l}\otimes\cdots\otimes D_{p_n}\\ &\xrightarrow{1\otimes\cdots\otimes m\otimes\cdots\otimes 1} D_{p_1}\otimes\cdots\otimes D_{p_\nu}\otimes D_{p_{\nu+1}}\otimes\cdots\otimes D_{p_n},\end{aligned}$$

with Δ standing for the diagonal map $D_{p_\nu+l}\to D_{p_\nu}\otimes D_l$ and m standing for the multiplication $D_l\otimes D_{p_{\nu+1}-l}\to D_{p_{\nu+1}}$.

For the skew shape λ/μ, we denote by $\operatorname{Rel}(\lambda/\mu)$ the module

$$\sum_{\nu=1}^{n-1}\sum_{l\geq t_\nu} D_{p_1}\otimes\cdots\otimes D_{p_\nu+l}\otimes D_{p_{\nu+1}-l}\otimes\cdots\otimes D_{p_n}$$

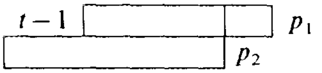
and by $\operatorname{Gen}(\lambda/\mu)$ the module $D_{p_1}\otimes\cdots\otimes D_{p_n}$.

Over the Schur algebra of degree Σp_ν, the tensor products of divided powers are projective (at least for rank $F\geq n$) and so the map □ is a projective presentation of $K_{\lambda/\mu}$. It is natural to ask whether $K_{\lambda/\mu}$ has a finite projective resolution, and to describe it if it exists. In [A-B.2], Akin and Buchsbaum proved that $K_{\lambda/\mu}$ has a finite projective resolution; in fact, they proved that the Schur algebra over a field or over the integers has finite global dimension. For skew shapes having two rows, they gave an explicit description of such resolutions. (It suffices to do this over the integers; the universality then carries the resolution to arbitrary commutative ground ring.)

In order to prove the existence of these resolutions, a class of shapes, far larger than the class of skew shapes, had to be introduced and studied. Before we define them precisely, let us indicate how they arise naturally.

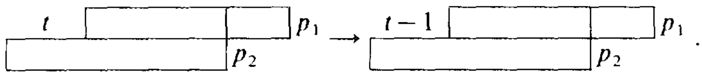
If we consider a 2-rowed skew shape

$$\lambda/\mu=(\lambda_1,\lambda_2)/(\mu_1,\mu_2):$$

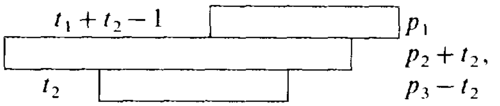


$$p_i=\lambda_i-\mu_i,\quad t=\mu_1-\mu_2+1,$$

we have a natural surjection from $K_{\lambda'/\mu'}$ onto $K_{\lambda/\mu}$, where $\lambda' = (\lambda_1, \lambda_2 - 1)$, $\mu' = (\mu_1, \mu_2 - 1)$:

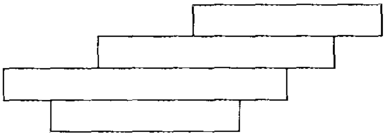


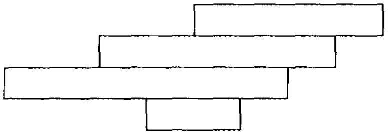
The kernel of this map is easily seen to be $K_{\lambda''}$, where λ'' is the partition $(\lambda_1 - \mu_2 + 1, \lambda_2 - \mu_1 - 1)$. However, if we consider the corresponding 3-rowed situation, taking $\lambda/\mu = (\lambda_1, \lambda_2, \lambda_3)/(\mu_1, \mu_2, \mu_3)$, $\lambda'/\mu' = (\lambda_1, \lambda_2, \lambda_3 - 1)/(\mu_1, \mu_2, \mu_3 - 1)$, and the natural surjection of $K_{\lambda'/\mu'}$ onto $K_{\lambda/\mu}$, the kernel is the Weyl module corresponding to a shape that looks like



where $p_i = \lambda_i - \mu_i$, $t_1 = \mu_1 - \mu_2 + 1$, $t_2 = \mu_2 - \mu_3 + 1$. This last shape "wants" to be a skew shape, but fails because the bottom row is too short.

For three rows, the above shape is the only "bad" type that can occur. For four rows, there are two types that arise, illustrated below:

(i)

;

(ii)

.

The first type occurs essentially as the kernel of a surjection between skew shapes, the second as the kernel of a surjection between shapes of type i.

More generally, the class of shapes, $\mathscr{I}$, that we consider (introduced in [A-B.1]) is defined by pairs λ/μ where $\lambda = (\lambda_1, ..., \lambda_n)$ is a partition, $\mu = (\mu_1, ..., \mu_n)$ is a sequence of non-negative integers such that

(a) $\mu_i \leqslant \lambda_i$ for $i = 1, ..., n$;

(b) $(\mu_1, ..., \mu_{n-1})$ is a partition;

(c) $\exists i$, $1 \leqslant i \leqslant n-2$, such that $\mu_{i+1} < \mu_n \leqslant \mu_i$ or μ is a partition, i.e., $\mu_n \leqslant \mu_{n-1}$;

(d) $\lambda_{n-1} - \lambda_n \geqslant n - i - 1$ or (if μ is a partition) $\lambda_{n-1} - \lambda_n \geqslant 0$.

(A slightly larger class of shapes, not imposing Condition d, has recently been introduced by [W].)

We say the n-rowed shape λ/μ is of type k if $\mu_{n-k+1} < \mu_n \leqslant \mu_{n-k}$ for $k = 2, \ldots, n-1$. We say it is of type 1 (or a skew shape) if $\mu_n \leqslant \mu_{n-1}$. A fundamental result is the following theorem, found in [A-B.1].

THEOREM 2.1. *Let λ/μ be an n-rowed shape of type k, and let λ'/μ' be the n-rowed shape defined by $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - 1)/(\mu_1, \ldots, \mu_{n-1}, \mu_n - 1)$. Then there is a natural surjection of $K_{\lambda'/\mu'}$ onto $K_{\lambda/\mu}$ whose kernel is*

(i) $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - 1)/(\mu_1, \ldots, \mu_{n-k-1}, \mu_n - 1, \mu_{n-k+1}, \ldots, \mu_{n-1}, \mu_{n-k})$ *if* $\mu_{n-k+1} < \mu_n - 1 < \mu_{n-k}$;

(ii) $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - 1)/(\mu_1, \ldots, \mu_{n-j-1}, \mu_n - 1, \mu_{n-j+1}, \ldots, \mu_{n-1}, \mu_{n-j})$ *if* $\mu_{n-j+1} < \mu_n - 1 \leqslant \mu_{n-j}$ *for some* $j < k$.

Note that λ'/μ' is of type $\leqslant k$. The kernel in case i is of type $k+1$, while in case ii it is of type j or $j+1$ (i.e., of type $\leqslant k$).

With Theorem 2.1, we get as an easy corollary a description of a presentation of $K_{\lambda/\mu}$ in terms of sums of tensor products of divided powers.

COROLLARY 2.2. *Let $\lambda/\mu = (\lambda_1, \ldots, \lambda_n)/(\mu_1, \ldots, \mu_n)$ be a shape in our class $\mathcal{I}$ of type $k \geqslant 2$. For each $i = 1, \ldots, k-1$, let $(\lambda/\mu)_i = (\lambda_1, \ldots, \lambda_{n-i}, \lambda_n - \mu_n + \mu_{n-i})/(\mu_1, \ldots, \mu_{n-i}, \mu_{n-i})$ and let $(\lambda/\mu)_k = (\lambda_1, \ldots, \lambda_{n-k}, \lambda_n)/(\mu_1, \ldots, \mu_{n-k}, \mu_n)$. For each $i = 1, \ldots, k$, $(\lambda/\mu)_i$ is a skew shape having $n-i+1$ rows. Up to permutation of factors,* $\mathrm{Gen}((\lambda/\mu)_i) \otimes D_{\lambda_{n-i}-\mu_{n-i}} \otimes \cdots \otimes D_{\lambda_{n-1}-\mu_{n-1}}$ *are all isomorphic to* $\mathrm{Gen}((\lambda/\mu)_1)$. *Denote by $\square_i$ the map of* $\mathrm{Rel}((\lambda/\mu)_i) \to \mathrm{Gen}((\lambda/\mu)_i)$, *and by $\square_i'$ the induced map*

$$\mathrm{Rel}((\lambda/\mu)_i) \otimes D_{\lambda_{n-i}-\mu_{n-i}} \otimes \cdots \otimes D_{\lambda_{n-1}-\mu_{n-1}}$$
$$\to \mathrm{Gen}((\lambda/\mu)_i) \otimes D_{\lambda_{n-i}-\mu_{n-i}} \otimes \cdots \otimes D_{\lambda_{n-1}-\mu_{n-1}} \approx \mathrm{Gen}((\lambda/\mu)_1).$$

Then the map $\sum_{i=1}^{k} \square_i' : \sum \mathrm{Rel}((\lambda/\mu)_i) \otimes D_{\lambda_{n-i}-\mu_{n-i}} \otimes \cdots \otimes D_{\lambda_{n-1}-\mu_{n-1}} \to \mathrm{Gen}((\lambda/\mu)_1)$ *is a presentation of $K_{\lambda/\mu}$.*

(The proof is a simple induction on k and $\mu_n - \mu_{n-k+1}$, using Theorem 2.1.)

Having this "universal" presentation of $K_{\lambda/\mu}$, we see that $K_{\lambda/\mu}$ commutates with a change of base ring. Since, over the integers, these modules are free (being subgroups of free Abelian groups), we immediately have the following corollary.

COROLLARY 2.3. *Let λ/μ be a shape in our class $\mathcal{I}$. Then $K_{\lambda/\mu}$ is universally free.*

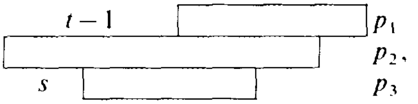
3. RESOLUTION OF SOME BAD SHAPES BY BETTER SHAPES

In this section, we show that n-rowed shapes in $\mathcal{S}$ of type $n-1$ can be "resolved" in terms of shapes in $\mathcal{S}$ of type $\leqslant n-2$. We then take up the problem of resolving shapes of type $n-2$ in terms of those of type $\leqslant n-3$, and so on.

Although we do not prove it here, it is relatively easy to see that an n-rowed shape of type $n-k$ can be resolved by n-rowed shapes of type $\leqslant n-k-1$ if and only if a $(k+2)$-rowed shape of type 2 can be resolved by $(k+2)$-rowed shapes of type $\leqslant 1$ (i.e., by skew shapes).

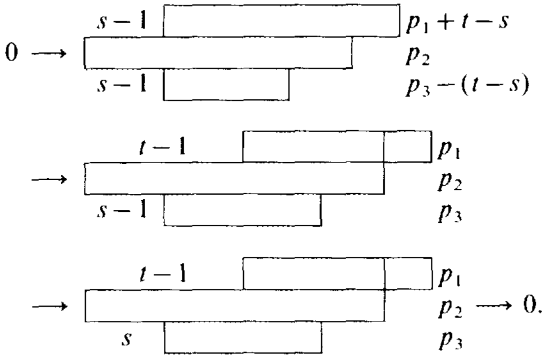
Let us therefore look at 3-rowed shapes of type 2. Such a shape may be described schematically by the diagram

(Σ)



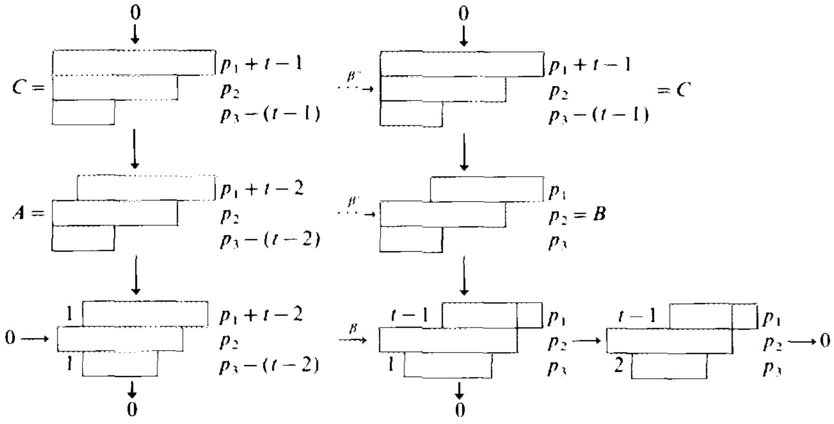
where $0<s<t$. By Theorem 2.1, we have the exact sequence:

(*)



(We write the sequence of shapes instead of the corresponding Weyl or Schur modules, as this makes what is going on a little clearer.)

If $s=1$, then $s-1=0$ and the left and middle terms of (*) are skew shapes, i.e., of type 1, so we have resolved our original shape of type 2 by skew shapes. We could now proceed by induction on s, as we did in [A-B.1], to describe resolutions of 2-rowed skew shapes, but it might be helpful to look first at the next step in this induction. Suppose, then, that our shape (Σ) has $s=2$. Then in our sequence (*) above our left and middle shapes have $s=1$, so that they can be resolved by skew shapes. We thus get a diagram



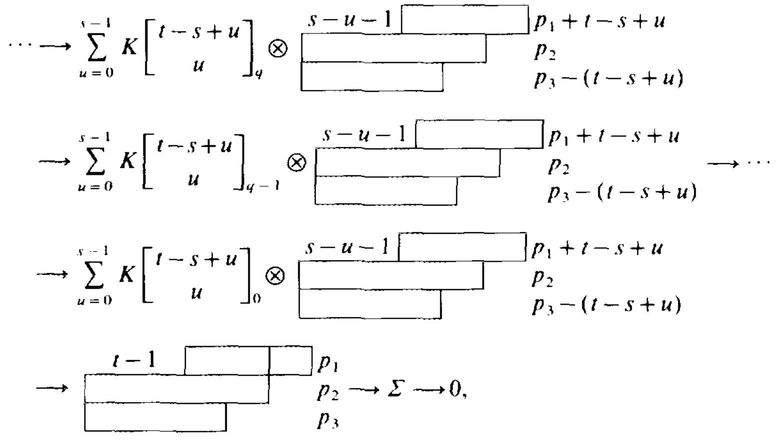
with exact bottom row (i.e., (*)) and exact columns.

The map β is the map induced by the map on the generators which diagonalizes the top row, D_{p_1+t-2}, to $D_{p_1}\otimes D_{t-2}$ and then multiplies D_{t-2} with the third row, $D_{p_3-(t-2)}$, to get D_{p_3}. Using the same map on the generators of A to the generators of B, one sees that a well-defined map, β', is induced between shapes A and B, and that β' clearly covers β. The map β'' is simply multiplication by $t-1$. We thus get the resolution of our shape Σ by skew shapes:

$$0 \to C \to A \oplus C \to B \to \Sigma \to 0.$$

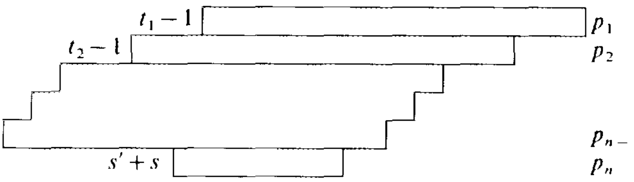
The main point of this discussion is that the map β admits an obvious lifting to the map $\beta'\colon A \to B$. This liftability of the map β in (*) is what makes the general inductive step work, and we can describe the resolution of Σ (for arbitrary s) in terms of shapes as

(R$_3$)



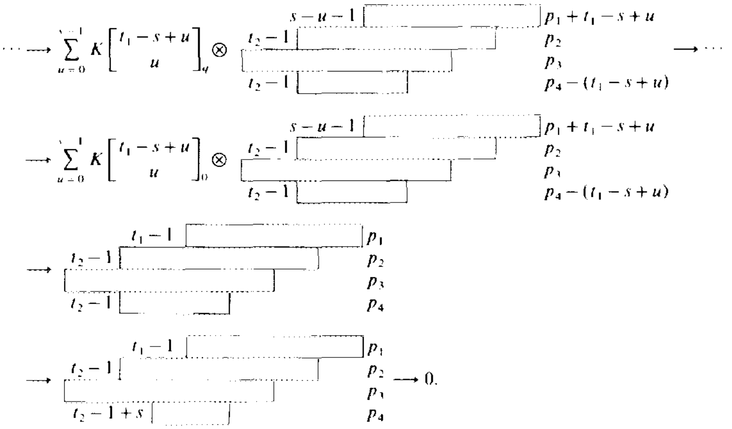
where $K[{}^{t-s+u}_{\ \ u}]$ is the arithmetic Koszul complex described in [A-B.1] and the boundary map is very much the same as that in the 2-rowed resolution of [A-B.1] except that the "polarization" is on the first and third rows rather than on the first and second.

To transcribe the resolution (R) to the n-rowed case, where Σ is the diagram



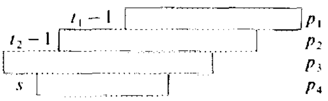
where $s'=\sum_{k=2}^{n-2} t_k-(n-3)$ and $0<s<t_1$, we simply let $t=t_1$ and essentially forget (i.e., just carry along without change of position) the rows $2, ..., n-1$. For example for $n=4$ we get

(R$_4$)



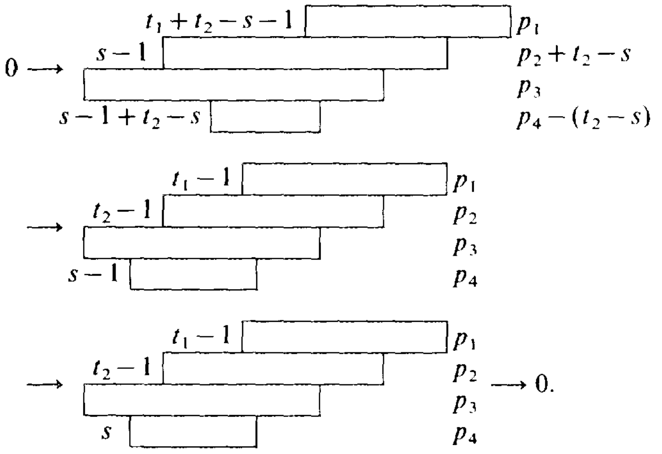
The second and third rows are rigid, while all the action occurs between the first and fourth rows: adding and subtracting boxes, and sliding the bottom row to the left so that it is flush with the second row. That polarization between the first and nth rows is a well-defined map on these shapes is easy to prove, using the presentation of these shapes given in Corollary 2.2.

Now what happens when we consider shapes of type $n-2$? In this case, we look at 4-rowed shapes of type 2,



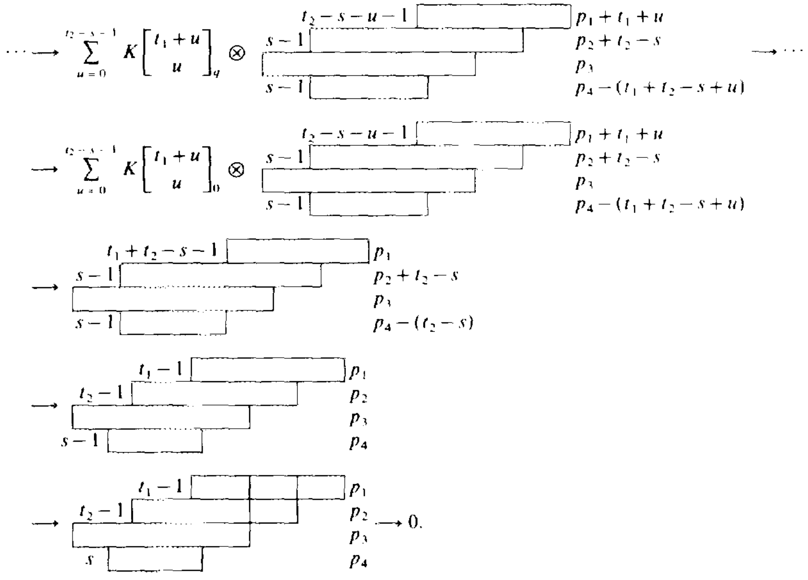
where $0 < s < t_2$, and ask if they can be resolved by skew shapes. As before, we use the fundamental exact sequence guaranteed us by Theorem 2.1:

(**)



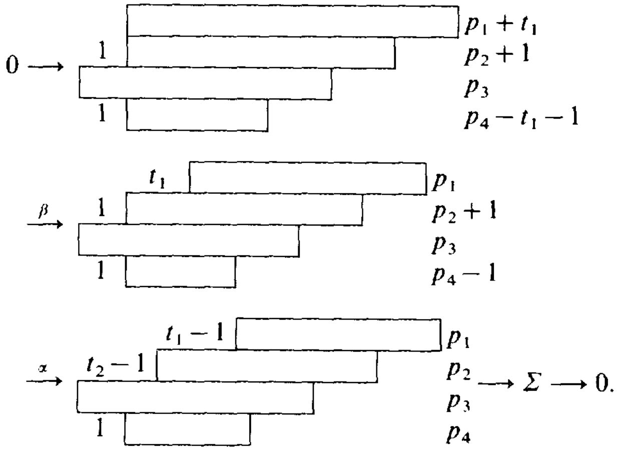
The middle term is also of type $\leqslant 2$, while the left term is of type 3. Applying our previous observations to this left term, i.e., splicing on the resolution, (R_4), of this term, we have

(E)



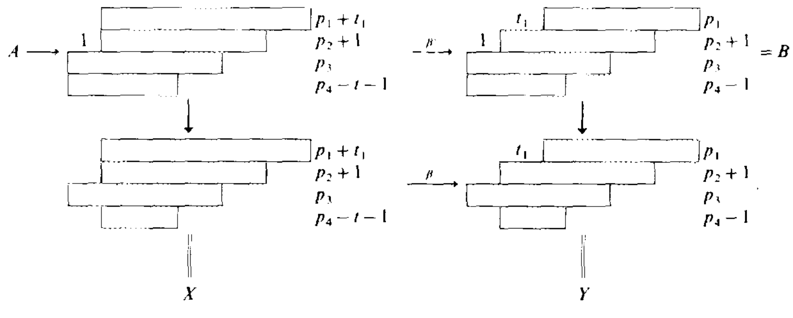
When $s = 1$, all the terms other than the one we started with are skew shapes, so that we have a resolution of our shape of type 2 in terms of skew shapes, as desired. Repeating the procedure that worked in the previous case, we examine what happens when $s=2$. To avoid having to work with large diagrams, let us assume that $s=2$ and $t_2=3$. In that case, our sequence (E) is conveniently short:

(E′)



Note that all the terms but Σ in (E') are of type 2 with $s=1$ (Σ has $s=2$). We have seen that shapes of type 2 with $s=1$ have resolutions in terms of skew shapes, so that we may resolve the terms of (E′) (all but Σ) by skew shapes and, as in our previous discussion, ask if we can cover (or lift) the maps α and β to maps between the skew-shape resolutions. The map α presents no difficulties, so we restrict our attention to the map β. The crucial question is, "Can we find a map β' as indicated by the dotted arrow below so that the diagram can be commutative?"

(D)



The vertical maps in the diagram (D) are induced by the identity maps on the generators of the modules. The map β is induced by "polarizing" the generators of the first row of X to the generators of the fourth row of Y.

If a similar polarization on the generators of A would induce a map from A to B, that clearly would be a lifting of β and we would be done. We show that simple-minded approach does not work. Moreover, we produce an example in Section 5 to show that one cannot perturb that polarization to produce *any* lifting of the map β. In order to follow this example, we review in the next section a few basic facts that we use.

4. WEIGHT SUBMODULES

In [A-B.2] it has been shown that if M is a polynomial representation of GL_n of degree d and A_d is the Schur algebra of degree d, then for all sequences of non-negative integers $(a_1, \ldots, a_n)$ of degree d (i.e., $a_1 + \cdots + a_n = d$), $Da_1 \otimes \cdots \otimes Da_n$ is A_d-projective and $\mathrm{Hom}_{A_d}(Da_1 \otimes \cdots \otimes Da_n, M)$ is the weight submodule of M corresponding to the weight $(a_1, \ldots, a_n)$. In particular, if M is the Weyl module of (skew) shape λ/μ, then M has a basis consisting of standard tableaux and the $(a_1, \ldots, a_n)$-weight submodule is a free abelian group having as a basis all standard tableaux of content $(a_1, \ldots, a_n)$. For example, if we take $M = K_{(4,1)}$ and $(a_1, a_2, \ldots, a_n) = (3, 2, 0, \ldots, 0)$, then $\mathrm{Hom}_{A_5}(D_3 \otimes D_2, K_{(4,1)})$ is free of rank 1, having as $\mathbb{Z}$-basis the standard tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} .$$

(Remember that a standard tableauy for Weyl modules is one with weakly increasing rows and strictly increasing columns.) The element of $\mathrm{Hom}_{A_5}(D_3 \otimes D_2, K_{(4,1)})$ corresponding to this tableau is the composite

$$D_3 \otimes D_2 \xrightarrow{1 \otimes \Delta} D_3 \otimes D_1 \otimes D_1 \xrightarrow{m \otimes 1} D_4 \otimes D_1 \xrightarrow{d'_{(4,1)}} K_{(4,1)},$$

where Δ is the diagonal, m is multiplication, and $d'_{(4,1)}$ is the Weyl map.

Let us return now to the question raised in Section 3 about lifting the map β in diagram (D). All the modules considered are representations of degree $p_1 + p_2 + p_3 + p_4 = q$, so all maps are Aq-maps (where Aq is the appropriate Schur algebra). Let $Y_1 \to Y_0 \xrightarrow{d'_A} A \to 0$ be a projective presentation of the Weyl module of shape A (we write A for K_A, X for K_X, etc.) where $Y_0 = D_{p_1+t_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-t_1-1}$. Then the diagram

$$\begin{tikzcd}
Y_0 \arrow[r, "d'_A"] \arrow[d, equal] & A \arrow[d] \\
Y_0 \arrow[r, "d'_X"] & X
\end{tikzcd}$$

is commutative, where the d' maps are the indicated Weyl maps. (In fact, the map $A \to X$ is the one induced by the identity map on Y_0.) The map $\beta \in \operatorname{Hom} Aq(X, Y)$ goes, by the inclusion

$$\operatorname{Hom}_{Aq}(X, Y) \to \operatorname{Hom}_{Aq}(Y_0, Y),$$

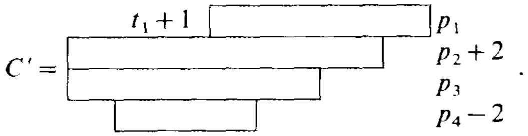
to the composite

$$\begin{aligned} &D_{p_1+t_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-t_1-1} \\ &\quad \xrightarrow{\Delta \otimes 1} D_{p_1} \otimes D_{t_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-t_1-1} \\ &\quad \xrightarrow{T} D_{p_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{t_1} \otimes D_{p_4-t_1-1} \\ &\quad \xrightarrow{1 \otimes m} D_{p_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-1} \xrightarrow{d'_Y} Y, \end{aligned}$$

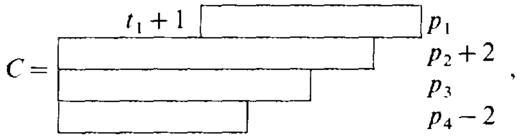
where Δ and m are diagonalization and multiplication, and T is the isomorphism switching D_{t_1} past D_{p_2+1} and D_{p_3}. Let us call this map τ_0.

To find a $\beta' \in \operatorname{Hom}_{Aq}(A, B)$ which lifts β is clearly equivalent to finding a map $\tau \in \operatorname{Hom}_{Aq}(Y_0, B)$ such that (a) τ goes to zero in $\operatorname{Hom}_{Aq}(Y_1, B)$ and (b) τ goes to τ_0 under the map $\operatorname{Hom}_{Aq}(Y_0, B) \to \operatorname{Hom}_{Aq}(Y_0, Y)$ induced by the map $B \to Y$.

Consider the kernel, C', of the map $B \to Y$. By Theorem 2.1, we have



Therefore, if we let C be



we have the natural surjection of C onto C' and the exact sequence

$$C \to B \to Y \to 0.$$

Because Y_0 is projective, the sequence

$$\operatorname{Hom}_{Aq}(Y_0, C) \to \operatorname{Hom}_{Aq}(Y_0, B) \xrightarrow{\Delta \otimes 1} \operatorname{Hom}_{Aq}(Y_0, Y) \to 0$$

is exact.

A simple diagram chase shows that a map $\tau \in \mathrm{Hom}_{Aq}(Y_0, B)$ satisfying conditions a and b exists if and only if, given a map $\tau_1 \in \mathrm{Hom}_{Aq}(Y_0, B)$ satisfying condition b, there is a map $\alpha \in \mathrm{Hom}_{Aq}(Y_0, C)$ such that the image of α in $\mathrm{Hom}_{Aq}(Y_1, B)$ is equal to the image of τ_1 in $\mathrm{Hom}_{Aq}(Y_1, B)$.

Now the map $\tau_1 \in \mathrm{Hom}_{Aq}(Y_0, B)$ defined by

$$\begin{aligned}
&D_{p_1+t_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-t_1-1}\\
&\quad \xrightarrow{\Delta \otimes 1} D_{p_1} \otimes D_{t_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-t_1-1}\\
&\quad \xrightarrow{T} D_{p_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{t_1} \otimes D_{p_4-t_1-1}\\
&\quad \xrightarrow{1 \otimes m} D_{p_1} \otimes D_{p_2+1} \otimes D_{p_3} \otimes D_{p_4-1} \xrightarrow{d'_B} B,
\end{aligned}$$

where the notation is as above, clearly satisfies condition b, and its image in $\mathrm{Hom}_{Aq}(Y_1, B)$ is easy to calculate. First of all, we know that we may take Y_1 to be the direct sum of the three modules:

$$Y_{11} : \oplus \sum_{l>0} D_{p_1+t_1+l} \otimes D_{p_2+1-l} \otimes D_{p_3} \otimes D_{p_4-t_1-1}$$

$$Y_{12} : \oplus \sum_{l>1} D_{p_1+t_1} \otimes D_{p_2+1+l} \otimes D_{p_3-l} \otimes D_{p_4-t_1-1}$$

$$Y_{13} : \oplus \sum_{l>0} D_{p_1+t_1} \otimes D_{p_2+1} \otimes D_{p_3+l} \otimes D_{p_4-t_1-1-l}.$$

The map $Y_1 \to Y_0 \xrightarrow{\tau_1} B$ is easily seen to carry Y_{12} and Y_{13} to zero. The only summand of Y_{11} not carried to zero is $D_{p_1+t_1+1} \otimes D_{p_2} \otimes D_{p_3} \otimes D_{p_4-t_1-1}$. Thus, to calculate the composition $Y_1 \to Y_0 \xrightarrow{\tau_1} B$, it suffices to see what happens to the element $1^{(p_1+t_1+1)} \otimes 2^{(p_2)} \otimes 3^{(p_3)} \otimes 4^{(p_4-t_1-1)}$ under this map. (We are writing $i^{(k)}$ of $x_i^{(k)}$, where $\{x_1, ..., x_n\}$ is a basis for our underlying free $\mathbb{Z}$-modules F.) The map $Y_1 \to Y_0$ takes this element to $1^{(p_1+t_1)} \otimes 1^{(1)} \otimes 2^{(p_2)} \otimes 3^{(p_3)} \otimes 4^{(p_4-t_1-1)}$ in Y_0, and τ_1 takes this to $d'_B(1^{(p_1)} \otimes 12^{(p_2)} \otimes 3^{(p_3)} \otimes 1^{(t_1)}4^{(p_4-t_1-1)})$. Simple application of the straightening rule quickly gives the result that

$$\begin{aligned}
&d'_B(1^{(p_1)} \otimes 12^{(p_2)} \otimes 3^{(p_3)} \otimes 1^{(t_1)}4^{(p_4-t_1-1)})\\
&\qquad = \pm d'_B(1^{(p_1)} \otimes 1^{(t_1)}2^{(p_2-t_1+1)} \otimes 12^{(t_1-1)}3^{(p_3-t_1)} \otimes 3^{(t_1)}4^{(p_4-t_1-1)}).
\end{aligned}$$

This latter is a standard tableau in B and part of a basis for $\mathrm{Hom}_{Aq}(Y_1, B)$. Thus this element is definitely not zero in $\mathrm{Hom}_{Aq}(Y_1, B)$. In order to solve our lifting problem, then, we must look at the map $\mathrm{Hom}_{Aq}(Y_0, C) \to \mathrm{Hom}_{Aq}(Y_1, B)$ and we whether its image contains the above element. In the next section, we give an example to show that the image does not always do so.

5. THE EXAMPLE

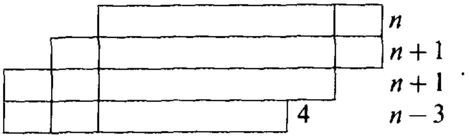
To construct our example, we have to choose the shape, Σ, of the sequence (E′) of Section 3, and then describe the modules A, B, C of Section 4, as well as the bases of the appropriate weight submodules: $\operatorname{Hom}_{A_q}(Y_0, C)$ and $\operatorname{Hom}_{A_q}(Y_1, B)$. We must also calculate the matrix of the map $\operatorname{Hom}_{A_q}(Y_0, C) \to \operatorname{Hom}_{A_q}(Y_1, B)$ to see if the element τ_1 of Section 4 is in the image.

For Σ, we take the shape

.

Thus, we have $p_1 = p_2$, $t_1 = 1$, $p_3 = p_1 + 1$, $p_4 = p_1 - 2$, and $t_2 = 3$, $s = 2$. Let us set $n = p_1 + 1$. Then C is the shape

and B is the shape



.

Our modules Y_0 and Y_1 are

$$Y_0 = D_{n+1} \otimes D_{n+1} \otimes D_{n+1} \otimes D_{n-4}$$

$$\begin{aligned} Y_1 = {} & \sum_{l \geqslant 0} D_{n+2+l} \otimes D_{n-l} \otimes D_{n+1} \otimes D_{n-4} \\ & \oplus \sum_{l \geqslant 0} D_{n+1} \otimes D_{n+3+l} \otimes D_{n-1-l} \otimes D_{n-4} \\ & \oplus \sum_{l \geqslant 0} D_{n+1} \otimes D_{n+1} \otimes D_{n+2+l} \otimes D_{n-5-l}, \end{aligned}$$

and our element $\tau_1 \in \operatorname{Hom}_{A_q}(Y_1, B)$ is the tableau

$$\begin{array}{|c|c|c|c|l}
\cline{3-4}
\multicolumn{2}{c|}{} & 1 \cdot \cdot \cdot \cdot \cdot \cdot \cdot 1 & 1 & n \\
\cline{2-4}
\multicolumn{1}{c|}{} & 1 & 2 \cdot \cdot \cdot \cdot \cdot \cdot \cdot 2 & 2 & n+1 \\
\hline
1 & 3 & 3 \cdot \cdot \cdot \cdot \cdot \cdot \cdot 3 & \multicolumn{1}{c}{} & n+1 \\
\hline
3 & 4 & 4 \cdot \cdot \cdot \cdot 4 \,|\, 4 & \multicolumn{1}{c}{} & n-3 \\
\cline{1-3}
\end{array}.$$

We agree to write this tableau as

$$1^{(n)} \mid 1\ 2^{(n)} \mid 1\ 3^{(n)} \mid 3\ 4^{(n-4)}.$$

In fact, we use this notation for tableaux in B and C in order to avoid printing cumbersome diagrams such as the one above.

To compute $\mathrm{Hom}_{Aq}(Y_0, C)$, we simply have to enumerate the standard tableaux in C with content $(1^{(n+)}, 2^{(n+1)}, 3^{(n+1)}, 4^{(n-4)})$. These are

β_1: $1^{(n)} \mid 1\ 2^{(n)}\ 4 \mid 2\ 3^{(n)} \mid 3\ 4^{(n-5)}$

β_2: $1^{(n)} \mid 1\ 2^{(n)}\ 3 \mid 2\ 3^{(n-1)}\ 4 \mid 3\ 4^{(n-5)}$

β_3: $1^{(n)} \mid 1\ 2^{(n+1)} \mid 3^{(n+1)} \mid 4^{(n-4)}$

β_4: $1^{(n)} \mid 1\ 2^{(n)}\ 3 \mid 2\ 3^{(n)} \mid 4^{(n-4)}$

β_5: $1^{(n-1)}\ 3 \mid 1^{(2)}\ 2^{(n-1)}\ 4 \mid 2^{(2)}\ 3^{(n-2)}\ 4 \mid 3^{(2)}\ 4^{(n-6)}$

β_6: $1^{(n-1)}\ 2 \mid 1^{(2)}\ 2^{(n-2)}\ 3\ 4 \mid 2^{(2)}\ 3^{(n-2)}\ 4 \mid 3^{(2)}\ 4^{(n-6)}$

β_7: $1^{(n-1)}\ 2 \mid 1^{(2)}\ 2^{(n-1)}\ 4 \mid 2\ 3^{(n)} \mid 3\ 4^{(n-5)}$

β_8: $1^{(n-1)}\ 3 \mid 1^{(2)}\ 2^{(n-1)}\ 4 \mid 2^{(2)}\ 3^{(n-1)} \mid 3\ 4^{(n-5)}$

β_9: $1^{(n-1)}\ 2 \mid 1^{(2)}\ 2^{(n-1)}\ 3 \mid 2\ 3^{(n-1)}\ 4 \mid 3\ 4^{(n-5)}$

β_{10}: $1^{(n-1)}\ 2 \mid 1^{(2)}\ 2^{(n-2)}\ 3^{(2)} \mid 2^{(2)}\ 3^{(n-2)}\ 4 \mid 3\ 4^{(n-5)}$

β_{11}: $1^{(n-1)}\ 2 \mid 1^{(2)}\ 2^{(n-2)}\ 3^{(2)} \mid 2^{(2)}\ 3^{(n-3)}\ 4^{(2)} \mid 3^{(2)}\ 4^{(n-6)}$

β_{12}: $1^{(n-1)}\ 2 \mid 1^{(2)}\ 2^{(n-1)}\ 3 \mid 2\ 3^{(n)} \mid 4^{(n-4)}$.

To compute $\mathrm{Hom}_{Aq}(Y_1, B)$, we note that although Y_1 is a direct sum of three different types of modules, the weight submodule of B corresponding to the middle type is zero, while the only non-zero weight submodules of B corresponding to the first and last types are those corresponding to $l = 0$. It is tedious but easy to check that the tableaux giving us a basis for $\mathrm{Hom}_{Aq}(Y_1, B)$ are

γ_1: $1^{(n)} \mid 1\ 2^{(n-1)}\ 4 \mid 1\ 2\ 3^{(n-1)} \mid 3^{(2)}\ 4^{(n-5)}$

γ_2: $1^{(n)} \mid 1\ 2^{(n-1)}\ 3 \mid 1\ 2\ 3^{(n-2)}\ 4 \mid 3^{(2)}\ 4^{(n-5)}$

γ_3: $1^{(n)} \mid 1\ 2^{(n-2)}\ 3^{(2)} \mid 1\ 2\ 3^{(n-2)}\ 4 \mid 2\ 3\ 4^{(n-5)}$

γ_4: $1^{(n)} \mid 1\ 2^{(n)} \mid 1\ 3^{(n)} \mid 3\ 4^{(n-4)} = \tau_1$

γ_5: $1^{(n)} \mid 1\ 2^{(n-1)}\ 3 \mid 1\ 2\ 3^{(n-1)} \mid 3\ 4^{(n-4)}$

γ_6: $1^{(n)} \mid 1\ 2^{(n-1)}\ 3 \mid 1\ 3^{(n)} \mid 2\ 4^{(n-4)}$

γ_7: $1^{(n)} \mid 1\ 2^{(n-1)}\ 3 \mid 2^{(2)}\ 3^{(n-1)} \mid 3^{(2)}\ 4^{(n-5)}$

γ_8: $1^{(n-1)}\ 2 \mid 1\ 2^{(n-1)}\ 3 \mid 1\ 2\ 3^{(n-1)} \mid 3^{(2)}\ 4^{(n-5)}$.

The matrix for the map $\mathrm{Hom}_{A_q}(Y_0, C) \to \mathrm{Hom}_{A_q}(Y_1, B)$ with respect to the bases $\{\beta_i\}$ and $\{\gamma_j\}$ can be calculated in a very straightforward, but complicated, way.

Remember that each basis element (i.e., tableau) β_i corresponds to a map from Y_0 to C. For instance, the tableau β_1 is the map from Y_0 to C which sends an element $x \otimes y \otimes z \otimes w$ of Y_0 to the image in the Weyl module C of the element in $\mathrm{Gen}(C)$ which is $\Sigma x(n) \otimes x'(1)\, y(n)\, w(1) \otimes y'(1)\, z(n) \otimes z'(1)\, w'(n-5)$ where $\Sigma x(n) \otimes x'(1)$ is the diagonal of x in $D_n \otimes D_1$, $\Sigma y(n) \otimes y'(1)$ is the diagonal of y in $D_n \otimes D_1$, $\Sigma w(1) \otimes w'(n-5)$ is that of w in $D_1 \otimes D_{n-5}$, and $\Sigma z(n) \otimes z'(1)$ is that of z in $D_n \otimes D_1$. Thus the image of β_1 in $\mathrm{Hom}_{A_q}(Y_1, B)$ is the composition

$$Y_1 \to Y_0 \xrightarrow{\beta_1} C \to B.$$

It clearly suffices to see what this composition does to the elements

$$1^{(n+2)} \otimes 2^{(n)} \otimes 3^{(n+1)} \otimes 4^{(n-4)}$$

and

$$1^{(n+1)} \otimes 2^{(n+1)} \otimes 3^{(n+2)} \otimes 4^{(n-5)} \in Y_1,$$

since these are the only weights that survive in B.

Applying straightening to the result of this composite map, one sees that the element β_1 is carried to $-\gamma_1 - (n-4)\gamma_4$. Carrying out this procedure on each β_i, we obtain the following matrix:

$$(\mathrm{M}) = \left(\begin{matrix}
-1 & 0 & 0 & 0 & (n-5)(2n-3) & (n-5)(n^2-6n+7) & 3(n-2) \\
0 & -1 & 0 & 0 & (n-5)(2n+1) & -2n(n-5) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(n-4) & -n(n-4) & -2 & n+1 & (n-4)(n-5) & 2(n-4)(n-5) & n(n-4) \\
0 & 0 & 0 & 1 & 2(n-4)(n-5) & (n-3)(n-4)(n-5) & 0 \\
0 & n-4 & 0 & -1 & -(n-4)(n-5) & -(n-1)(n-4)(n-5) & 0
\end{matrix}\right.$$

$$\left.\begin{matrix}
-(n-1) & 0 & 0 & 0 & 0 \\
n-5 & n-2 & -(n-3)^2 & -2(n-3)(n-5) & 0 \\
-2 & 0 & -2 & 2(n-5) & 0 \\
2(n-1)(n-4) & (n-4)(n^2+3n-3) & -(n-1)(n-4)^2 & \binom{n-4}{2}(n^2-n+2) & -n(n+1) \\
(n-2)(n-4) & -3(n-4)(n-1) & 2(n-1)(n-4) & \binom{n-4}{2}(n^2-9n+12) & n-2 \\
(n-1)(n-4) & -(n-2)(n-4) & (n-1)^2(n-4) & -\binom{n-4}{2}(n-1)n & n-2
\end{matrix}\right)$$

We have left out the last two rows of the matrix, i.e., the coefficient of γ_7 and γ_8, as they are not needed to reach the conclusion that τ_1, or γ_4, is not in the image of the map.

When we reduce the above matrix modulo 2, we get

$$\begin{matrix} 1 & 0 & 0 & 0 & n+1 & n+1 & n & n+1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & n+1 & 0 & 0 & n+1 & n & n+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n & n & 0 & n+1 & 0 & 0 & n & 0 & n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & n & 0 & 0 & 0 & n \\ 0 & n & 0 & 1 & 0 & 0 & 0 & 0 & n & 0 & 0 & n. \end{matrix}$$

For n even we get

$$\begin{matrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0, \end{matrix}$$

while for n odd we get

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1. \end{matrix}$$

In either case, we see that τ_1 cannot be in the image when we reduce mod 2, so it cannot be in the image of our integral matrix, (M).

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