ABSTRACT

APPROACHES TO RESOLUTIONS OF WEYL MODULES

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In a paper with Kaan Akin [Adv. in Math. V 72 No. 2 (1988) 171-210], we succeeded in writing down an explicit resolution of Weyl modules corresponding to two-rowed skew shapes. (By resolution I mean a projective resolution over the Schur algebra of appropriate degree.) To do this, we introduced an 'arithmetic Koszul complex' and used certain fundamental exact sequences together with convenient mapping cone properties of these complexes. The attempt to extend these techniques to obtain resolutions of skew shapes with a larger number of rows led to calculations of almost astronomical proportions and, except for a very small number of special cases, the program pretty much came to a halt.

In the fall of 1991, G-C Rota and I decided to look at the problem of resolving Weyl modules by means of 'letter-place' techniques. It soon became clear that all of the standard maps used in the two-rowed resolutions were polarizations (of letters), and that if we introduced letter-place alphabets, the maps could also be seen to be place polarizations. Furthermore, replacing the Koszul complex by an appropriate Bar Complex simplified the description of the terms of the complex enormously. (Earlier, Akin and I had seen that the Koszul complex could be replaced by the Bar Complex, but we hadn't seen any particular benefits deriving from this alternate description.) Once we perceived that the maps we were dealing with were polarizations, it was reasonable to look at some of the calculations that had been done in the three-rowed case as reflecting Capelli-like identities on polarization operators. In relatively short order it was possible to establish identities of the following sort:
I) \( \partial_{3,2}^{(a)} \partial_{2,1}^{(b)} = \prod_u \partial_{2,1}^{(b\mu \nu)} \partial_{3,2}^{(a)} \partial_{3,3}^{(a\mu \nu)} \)

II) \( \partial_{3,2}^{(a)} \partial_{3,1}^{(b)} = \partial_{3,1}^{(b\mu \nu)} \partial_{3,2}^{(a)} \)

III) \( \partial_{2,1}^{(a)} \partial_{3,1}^{(b)} = \partial_{2,1}^{(b\mu \nu)} \partial_{3,1}^{(a)} \)

IV) \( \partial_{2,1}^{(b\mu \nu)} \partial_{3,2}^{(a)} = \sum_u (\prod^u) \partial_{3,2}^{(a\mu \nu)} \partial_{2,1}^{(b\mu \nu)} \partial_{3,1}^{(a)} \).

(The subscripts on the polarization operators indicate the place polarizations, e.g., the subscript 3, 2 indicates that the polarization is from the second to the third place. The exponents in parentheses are the indicated divided powers of these operators.)

Using these identities, the special cases that had previously been considered (at least for three rows) were now almost transparently clear. By ringing suitable changes on I) - IV) above, we obtained:

V) \( \partial_{3,2}^{(a)} \partial_{3,1}^{(b)} = \prod_u (\prod^u) \partial_{3,2}^{(a\mu \nu)} \partial_{3,1}^{(b\mu \nu)} \partial_{3,1}^{(a\mu \nu)} + (\prod^u) \partial_{3,1}^{(b\mu \nu)} \partial_{3,2}^{(a\mu \nu)} \partial_{3,2}^{(a\mu \nu)} \)

VI) \( \partial_{3,1}^{(a)} \partial_{2,1}^{(b)} = \prod_u (\prod^u) \partial_{3,2}^{(a\mu \nu)} \partial_{3,1}^{(a\mu \nu)} \partial_{2,1}^{(b\mu \nu)} \partial_{2,1}^{(b\mu \nu)} + (\prod^u) \partial_{2,1}^{(a\mu \nu)} \partial_{3,2}^{(a\mu \nu)} \partial_{3,2}^{(a\mu \nu)} \).

These identities helped push us to obtain some extensions of the class of skew shapes for which we could write, in a conceptually clear way, a projective resolution.

"Conceptual clarity" is not easy to define (and I won't attempt to in this abstract), but it was in the pursuit of such clarity that Rota and I were led to the notion of a Generalized Bar Complex [Proc. Natl. Acad. Sci. USA, vol 91, pp 4115-4119, (May, 1994)], and used this notion to describe certain resolutions. The idea involves taking the free product of an algebra of 'formal' polarization operators with the exterior algebra of 'separator variables', and creating a boundary map by polarizing certain of the separators to 1. In this free product we introduce relations suggested by I) to VI) above, such as:
I) \[ Z_{3,2}^{(a)} Z_{2,1}^{(b)} x = \bigotimes_{u \in \mathbb{Z}} Z_{2,1}^{(b)u} x Z_{3,2}^{(a)u} \delta_{3,3}^{(u)} \]

II) \[ Z_{3,2}^{(a)} x = x Z_{3,2}^{(a)} \]

III) \[ \delta_{3,1}^{(a)} x = x \delta_{3,1}^{(a)}, \quad \delta_{3,1}^{(a)} Z_{2,1}^{(b)} = Z_{2,1}^{(b)} \delta_{3,1}^{(a)} \]

IV) \[ Z_{3,1}^{(a)} (Z_{2,1}^{(b)} x \cdots x Z_{2,1}^{(b)} x v) = (\bigotimes_{u \in \mathbb{Z}} Z_{2,1}^{(b)u} x Z_{3,2}^{(a)u} (Z_{2,1}^{(b)} x \cdots x Z_{2,1}^{(b)} x v) + \]

V) \[ Z_{3,1}^{(a)} (Z_{3,2}^{(h)} y \cdots y Z_{3,2}^{(h)} y Z_{2,1}^{(c)} x \cdots x Z_{2,1}^{(c)} x v) = \]

VI) \[ Z_{3,2}^{(a)} (Z_{3,1}^{(c)} z Z_{2,1}^{(c)} x \cdots x Z_{2,1}^{(c)} x v) = (\bigotimes_{u \in \mathbb{Z}} Z_{2,1}^{(c)u} x Z_{3,2}^{(a)u} Z_{2,1}^{(c)u} x \cdots x Z_{2,1}^{(c)u} x v) + \]

Notice that the 'formal' operators (denoted by Z) are distinguished from the non-formal ones because they satisfy different kinds of commutativity relations. For example, III tells us that the ordinary operator polarizing the first place to the third commutes with the formal polarization from the first to the second place. Relation IV, however, doesn't permit the formal counterpart to commute in the same way. (The letters x, y, and z indicate separator variables.)

We will show how considerations of this sort direct the program to find characteristic-free resolutions of Weyl modules corresponding not only to skew shapes, but to a much broader class of shapes. We will also show that the consideration of this broader class of shapes actually makes the problem for skew shapes easier to deal with, rather than harder. However, here I will write down only the resolutions of Weyl modules corresponding to n-rowed skew-shapes, and this will be done inductively, in the sense that we will assume that we know the resolutions for n-1 - rowed shapes and work from there.

First, a bit of notation. We denote by \((p_1, \cdots, p_n; \square_1, \cdots, \square_{[k]}\) the shape corresponding to the skew-partition \((\square_1, \square_2, \cdots, \square_n) / (\square_1, \square_2, \cdots, \square_n)_k\) where \(\square_n = 0, \square_k = \square_{[1]} + \cdots + \square_k\) for \(k = 1, \ldots, n-1\), and \(\square_k = p_k + \square_k\) for \(k = 1, \ldots, n\). By
\( Res(p_1, \cdots, p_n; \square, \cdots, \square) \) we will mean the resolution of the n-1 - rowed skew shape \((p_1, \cdots, p_n; \square, \cdots, \square)\). We then have:

\[
Res(p_1, \cdots, p_n) = Res(p_1, \cdots, p_n; \square, \cdots, \square) + Res(p_1, \cdots, p_n; \square, \cdots, \square, \square) \]

\[
\sum_{l_{ij} \geq 0} \left( \sum_{l_{ij} \geq 0} \left( Z_{n,n}^{(l_{ij}+1)} Z_{n,n}^{(l_{ij})} \right) \right) Res(p_1, \cdots, p_n; \square, \cdots, \square, \square) + Res(p_1, \cdots, p_n; \square, \cdots, \square, \square, \square) \]

\[
\sum_{l_{ij} \geq 0} \left( \sum_{l_{ij} \geq 0} \left( Z_{n,n}^{(l_{ij}+1)} Z_{n,n}^{(l_{ij})} \right) \right) \cdots \left( Z_{n,n}^{(l_{ij})} \right) \]

\[
Res(p_1 + \square, \cdots, p_n + \square; \square, \cdots, \square, \square, \square, \square) \]

where \( \square_j = \square_j + l_j + 1 \) for \( j = 1, \cdots, n \).

The notation \( Z_{n,n}^{(l_{ij}+1)} Z_{n,n}^{(l_{ij})} \) stands for a certain homogeneous strand of a specified Bar Complex.

Those who are familiar with Lascoux' resolution in characteristic zero will notice a structural similarity between it and the resolution described above. Recall that Lascoux has described the terms and their placement in that resolution in terms of the lengths of the permutations corresponding to the determinantal expansion of the Jacobi-Trudi matrix for the Weyl module. If we let \( S_n \) denote the symmetric group on \( n \) letters, and \( \square S_n \) a permutation such that \( \square(n) = i \), then \( s \) can be written uniquely as a product: \( \square = (n, n \square 1)(n, n \square 2) \cdots (n, i) \square \), where \((n,j)\) stands for the transposition, and \( \square S_n \). (Notice that the length of \( s \) is \( n - i \) +length of \( \square \).) This provides us with a way of recovering the terms of the Lascoux resolution within the resolution described above.