LETTER-PLACE METHODS AND HOMOTOPY

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Dedicated to Gian-Carlo Rota

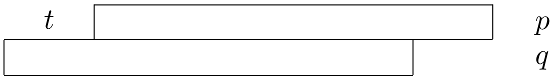
1. Introduction

In this talk I want to illustrate the use of letter-place methods in the construction of homotopies for resolutions of certain Weyl modules. I'll look at two cases: the resolutions of two-rowed skew-shapes (already in the literature [B-R 1]), and the resolutions of hooks (work in progress; only the first three stages of the homotopy appear here). The desire to construct such homotopies arose from the attempt to answer Rota's question: What do the syzygies of these modules look like? Since, in representation theory and combinatorics, the 'correct' form of an answer to such a question is in terms of a basis consisting of easily described tableaux, the double standard tableaux that appear in letter-place algebras lend themselves admirably to this topic.

Throughout this paper, we will be considering a fixed free module, F, over a commutative ring, and the representations of the general linear group of F. In general, we will omit the letter F when we write the divided powers, exterior powers, and other functors applied to F. When we speak of projective resolutions of Weyl modules, the ring over which we are taking these resolutions is the Schur algebra of appropriate degree. Since the resolutions we are describing are universal and not necessarily minimal, we may as well assume that we are always working over the ring of integers, and hence the Schur algebra in question is the integral Schur algebra.

2. The Two-rowed case

In this section we look at the general two-rowed skew-shape:



where the top row has p boxes, the second row has q boxes, and the protuberance of the second row to the left of the first is t. Expressed in terms of skew partitions this is simply $(p+t,q)/(t,0)$.

The Weyl module associated to this shape has the resolution:

$$0 \to M_{q-t} \xrightarrow{d_{q-t}} M_{q-t-1} \xrightarrow{d_{q-t-1}} \cdots \xrightarrow{d_3} M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0$$

where

$$\begin{aligned}
M_0 &= D_p \otimes D_q; \\
M_1 &= \sum_{l>t} Z_{2,1}^{(l)} x D_{p+l} \otimes D_{q-l}; \\
&\vdots \\
M_k &= \sum_{l_1>t} Z_{2,1}^{(l_1)} x Z_{2,1}^{(l_2)} x \cdots x Z_{2,1}^{(l_k)} x D_{p+|l|} \otimes D_{q-|l|} \\
&\vdots \\
M_{q-t} &= Z_{2,1}^{(t+1)} x Z_{2,1}^{(1)} x \cdots x Z_{2,1}^{(1)} x D_{p+q} \otimes D_0.
\end{aligned}$$

By D_r we mean the divided power of degree r of the underlying free module F. The symbol $Z_{2,1}^{(l)}$ stands for the divided power of degree l of the free generator $Z_{2,1}$, and the action of $Z_{2,1}^{(l)}$ on any term $D_u \otimes D_v$ is as place polarization of degree l from place one to place two. By $|l|$ we mean the sum $l_1 + \cdots + l_k$, and the symbol x stands for a separator variable (in the sense of [B-R 2]). All the indices l_i are assumed to be positive. The boundary maps are those of the Bar Complex as described in the same article. For example, using letter-place notation for the basis elements of $D_u \otimes D_v$, we can describe the boundary map $d_2 : M_2 \to M_1$ as follows:

$$d_2\left(Z_{2,1}^{(l_1)} x Z_{2,1}^{(l_2)} x \left[\begin{array}{c|cc} w & 1^{(p+l_1+l_2)} & 2^{(q_1)} \\ w' & 2^{(q_2)} & \end{array} \right] \right) =$$

$$Z_{2,1}^{(l_1)} x \binom{l_2+q_1}{q_1} \left[\begin{array}{c|cc} w & 1^{(p+l_1)} & 2^{(l_2+q_1)} \\ w' & 2^{(q_2)} & \end{array} \right] - \binom{l_1+l_2}{l_2} Z_{2,1}^{(l_1+l_2)} x \left[\begin{array}{c|cc} w & 1^{(p+l_1+l_2)} & 2^{(q_1)} \\ w' & 2^{(q_2)} & \end{array} \right].$$

As the letter-place notation implies, we are assuming that

$$\begin{aligned}
q_1 + q_2 &= q - l_1 - l_2; \\
q_2 &\leq p + l_1 + l_2.
\end{aligned}$$

(Of course, given that $p + t \geq q$, the second condition is always satisfied.)

In [B-R 2] we describe the homotopy maps $s_k : M_k \to M_{k+1}$, $k = 0, \cdots, q-t-1$ as follows:

$$s_0\left(\left[\begin{array}{c|cc} w & 1^{(p)} & 2^{(q_1)} \\ w' & 2^{(q_2)} & \end{array} \right] \right) = \begin{cases} Z_{2,1}^{(q_1)} x \left[\begin{array}{c|c} w & 1^{(p+q_1)} \\ w' & 2^{(q_2)} \end{array} \right] & \text{if } q_1 > t \\ 0 & \text{otherwise} \end{cases}$$

$$s_k\left(Z_{2,1}^{(l_1)} x Z_{2,1}^{(l_2)} x \cdots x Z_{2,1}^{(l_k)} x \left[\begin{array}{c|cc} w & 1^{(p+|l|||)} & 2^{(q_1)} \\ w' & 2^{(q_2)} & \end{array} \right] \right)$$
$$= \begin{cases} Z_{2,1}^{(l_1)} x Z_{2,1}^{(l_2)} x \cdots x Z_{2,1}^{(l_k)} x \left[\begin{array}{c|c} w & 1^{(p+|l|+q_1)} \\ w' & 2^{(q_2)} \end{array} \right] & \text{if } q_1 > 0 \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to see that this is a splitting contracting homotopy for the resolution. It's well-known that, if we have any free complex $\mathbf{X} = \{X_i; d_i\}$

with a splitting contracting homotopy $\{s_i\}$, if $\{x_{i\alpha}\}$ is the subset of the basis of X_i such that $s_i(x_{ia}) \neq 0$, and the set $\{s_i(x_{ia})\}$ is linearly independent, then the set $\{d_{i+1}s_i(x_{i\alpha})\}$ is a basis for the cycles in dimension i. Applying this to our case, we see that in dimension 0 the basis for the syzygies can be taken to be the set:

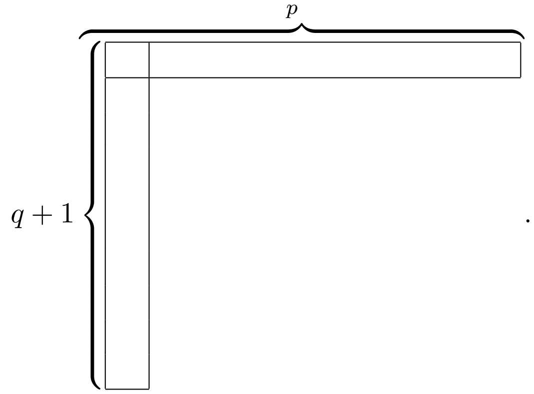
$$\left\{ \; Z_{2,1}^{(t+r)} x \left[\begin{array}{c|c} w & 1^{(p+t+r)} \\ w' & 2^{(q-t-r)} \end{array} \right]; \quad r > 0 \; \right\},$$

while in positive dimension, k, the basis can be taken to be the set:

$$\left\{ Z_{2,1}^{(t+r_1)} x Z_{2,1}^{(r_2)} x \cdots x Z_{2,1}^{(r_{k+1})} x \left[\begin{array}{c|c} w & 1^{(p+t+|r|)} \\ w' & 2^{(q-t-|r|)} \end{array} \right]; r_i > 0 \right\}.$$

3. The Hooks

In this section we will write down a resolution of the Weyl modules corresponding to the partitions known as 'hooks'. This kind of partition is of the form $(p, 1^{(q)})$, and corresponds to the picture



If one simply were looking for a resolution of this Weyl module, and not interested in describing a basis for the syzygies, one could proceed as follows.

We first write down the projective resolution of Λ^k for each k. Our contention is that it's the following:

$$\begin{aligned} \mathbf{X}^{(k)} \quad : \quad & D_k \xrightarrow{\partial} \cdots \xrightarrow{\partial} \sum_{i_j > 0} D_{i_1} \otimes \cdots \otimes D_{i_{k-p}} \xrightarrow{\partial} \\ & \sum_{i_j > 0} D_{i_1} \otimes \cdots \otimes D_{i_{k-p+1}} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underbrace{D_1 \otimes \cdots \otimes D_1}_{k} \end{aligned}$$

where the sum of the indices is always equal to k, and

$$\partial(a_1 \otimes \cdots \otimes a_{k-p}) = \sum_{j=1}^{k-p} (-1)^j a_1 \otimes \cdots \otimes \Delta'(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_{k-p}.$$

By $\Delta'(a_j)$ we mean:

$$\Delta'(a_j) = \sum_{0<l<i_j} a_j(l) \otimes a_j(i_j - l).$$

That this is a complex can be seen from the fact that Δ is coassociative.

To prove acyclicity, we define a filtration on this complex, and assume by induction that $\mathbf{X}^{(q)}$ is acycylic (i.e., has vanishing homology in positive dimensions, and has homology equal to Λ^q in dimension 0).

Define $F_p(\mathbf{X}^{(k)})$ to be the subcomplex of $\mathbf{X}^{(k)}$ consisting of those terms with $i_1 \leq p$. We have

$$0 \subset F_1(\mathbf{X}^{(k)}) \subset F_2(\mathbf{X}^{(k)}) \subset \cdots \subset F_k(\mathbf{X}^{(k)}) = \mathbf{X}^{(k)}.$$

Then

$$F_p(\mathbf{X}^{(k)})/F_{p-1}(\mathbf{X}^{(k)}) \approx D_p \otimes \mathbf{X}^{(k-p)}$$

with a dimension shift of $p-1$, i.e.,

$$F_p(\mathbf{X}^{(k)})/F_{p-1}(\mathbf{X}^{(k)}) \approx 0$$

in dimensions $0, \cdots, p-2$, while

$$F_p(\mathbf{X}^{(k)})/F_{p-1}(\mathbf{X}^{(k)}) \approx D_p \otimes \mathbf{X}_l^{(k-p)}$$

in dimensions $p-1+l$, for $l \geq 0$.

Therefore, the E^1 term of the spectral sequence (i.e., the homology of the E^0 term described above) is $D_p \otimes \Lambda^{k-p}$. Thus the E^2 term is just Λ^k, and standard arguments finish the proof.

Now we can write down the resolution of the hook, $K_{(p,1^q)}$. In terms of the preceding notation, it's simply

$$\mathbf{X}^{(p+q)}/F_{p-1}(\mathbf{X}^{(p+q)})$$

with a dimension shift of $p-1$.

To see this easily, we use induction on q, and the exact sequence:

$$0 \to K_{(p+1,1^{q-1})} \to D_p \otimes \Lambda^q \to K_{(p,1^q)} \to 0.$$

Without going into too much detail, we just state that the map from the resolution of $K_{(p+1,1^{q-1})}$, which is $\mathbf{X}^{(p+q)}/F_p(\mathbf{X}^{(p+q)})$, into $D_p \otimes \Lambda^{k-p}$ is induced by:

$$a_1 \otimes \cdots \otimes a_{k-p} \mapsto \sum a_1(p) \otimes a_1(i_1 - p) \otimes a_2 \otimes \cdots \otimes a_{k-p}.$$

(The notation $\sum a_1(p) \otimes a_1(i_1 - p)$ denotes the diagonalization of a_1 into its degree p and degree $i_1 - p$ components.)

With this, it's easy to see that the mapping cone is the indicated resolution of our hook, and we have succeeded in our first effort. The problem now is to set these resolutions up in letter-place terminology so that we will be in a position to start defining a splitting contracting homotopy. (I know of no way of describing such a homotopy without resorting to the letter-place basis. That no equivariant homotopy exists is, I believe, fairly obvious.)

In old-fashioned terms, we know that the modules that comprise the n-dimensional chains of the resolution of the hook are direct sums of modules of the form

$$(*) \quad D_{p-1+i_0} \otimes D_{i_1} \otimes \cdots \otimes D_{i_{q-n}}$$

with all $i_j > 0$ and $i_0 + \cdots + i_{q-n} = q + 1$.

What we want to do is prefix these terms with the appropriate operators and separators, and say which of the $q + 1$ places the identified terms occupy. The boundary operator will be the usual sum of the polarizations of the separators to 1, but to make sense of this boundary, we have to make explicit certain Capelli-like identities (which will be described soon). First, we specify which places the factors in the term $(*)$ occupy:

$$\begin{aligned}
&D_{p-1+i_0} \text{ is in place number } 1;\\
&D_{i_1} \text{ is in place number } i_0 + 1;\\
&D_{i_2} \text{ is in place number } i_0 + i_1 + 1;\\
&\vdots\\
&D_{i_j} \text{ is in place number } i_0 + i_1 + \cdots + i_{j-1} + 1;\\
&\vdots\\
&D_{i_{q-n}} \text{ is in place number } i_0 + i_1 + \cdots + i_{q-n-1} + 1.
\end{aligned}$$

(The sum $i_0 + i_1 + \cdots + i_{q-n-1} + 1$ is less than or equal to $q + 1$.) Of course this means that in all the other places we have D_0.

Next we define some operators:

For each integer $i \geq 1$, and each integer $k \geq 1$, define the element in the appropriate free-product algebra (see [B-R 2]):

$$Z(i;k) = Z^{(1)}_{k+i-1,k+i-2} x_{k+i-2} Z^{(2)}_{k+i-2,k+i-3} x_{k+i-3} \cdots x_{k+2} Z^{(i-2)}_{k+2,k+1} x_{k+1} Z^{(i-1)}_{k+1k} x_k.$$

We agree that if $i = 1$, then this element is just equal to the identity.

I claim that the term $(*)$ above is, in the resolution of the hook, preceded by:

$$Z(i_{q-n}; k_{q-n}) Z(i_{q-n-1}; k_{q-n-1}) \cdots Z(i_1; k_1) Z(i_0; k_0)$$

where $k_j = i_0 + i_1 + \cdots + i_{j-1} + 1$ for $j = 1, \cdots, q - n$, and $k_0 = 1$. (That this is indeed an operator of length n is easy to see.)

To make sense of the above boundary map, we introduce the Capelli-like identities mentioned above:

$$(C) \qquad \begin{array}{l} Z^{(a)}_{k+\beta,k} Z^{(b)}_{k,k-\alpha} x_{k-\alpha} = \sum_{b>u\geq 0} Z^{(b-u)}_{k,k-\alpha} x_{k-\alpha} Z^{(a-u)}_{k+\beta,k} Z^{(u)}_{k+\beta,k-\alpha};\\ Z^{(a)}_{k+\beta_1+\beta_2,k+\beta_2} Z^{(b)}_{k,k-\alpha} x_{k-\alpha} = Z^{(b)}_{k,k-\alpha} x_{k-\alpha} Z^{(a)}_{k+\beta_1+\beta_2,k+\beta_2}. \end{array}$$

With these identities, it is easy to see that we have reproduced the resolution in letter-place terms, although we haven't yet put the divided power terms into letter-place notation. To make the conversion to letter-place complete, and to make calculation with the boundary map easier to handle, we'll proceed as follows:

Let's suppose we have words, w, in letters, and places $1, 2, \cdots, q+1$. Then the terms we're looking at in dimension n correspond to sequences of places, $k_0, k_1 \cdots, k_{q-n}$, satisfying the following conditions:

$$\begin{aligned} k_0 &= 1, \\ k_j - k_{j-1} &= i_{j-1} > 0. \end{aligned}$$

We set $i_{q-n} = q + 2 - k_{q-n} > 0$. Then the typical term in dimension n is

$$\begin{aligned} &Z(i_{q-n}; k_{q-n}) Z(i_{q-n-1}; k_{q-n-1}) \cdots Z(i_1; k_1) \cdot \\ &Z(i_0; k_0)(w_0|k_0^{(p-1+i_0)})(w_1|k_1^{(i_1)}) \cdots (w_{q-n}|k_{q-n}^{(i_{q-n})}). \end{aligned}$$

Since this notation is a little obscure until one becomes used to it, it's probably worthwhile to look at a simple, but not altogether trivial, example. Suppose we start out with a term in $D_{p+3_0} \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q-3}$. In this case, we have the sequence $\{i_0, i_1, \cdots, i_{q-3}\}$, where $i_0 = 4$, $i_1 = 1, \cdots, i_{q-3} = 1$. This then gives rise to the sequence of 'places',
$\{k_0, k_1, \cdots, k_{q-3}\}$, where $k_0 = 1, k_1 = 5, k_2 = 6, \cdots, k_{q-3} = q+2$ and we see that a typical generator of $D_{p+3_0} \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q-3}$ is of the form

$$(w_0|1^{(p+3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)})$$

where we are using the integers to stand for the positive places. The operator prefix for this term is simply $Z(4; 1)$ since all the i_j are equal to 1 except for $j = 0$. Thus we have to compute the boundary of

$$Z_{4,3}^{(1)} x_3 Z_{3,2}^{(2)} x_2 Z_{2,1}^{(3)} x_1 (w_0|1^{(p+3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)}).$$

Consider

$$\begin{aligned} &Z_{4,3}^{(1)} x_3 Z_{3,2}^{(2)} x_2 Z_{2,1}^{(3)} x_1 (w_0|1^{(p+3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)}) \mapsto \\ &\quad Z_{4,3}^{(1)} x_3 Z_{3,2}^{(2)} x_2 (w_0|1^{(p)} 2^{(3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)}) - \\ &\quad Z_{4,3}^{(1)} x_3 Z_{3,2}^{(2)} Z_{2,1}^{(3)} x_1 (w_0|1^{(p+3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)}) + \\ &\quad Z_{4,3}^{(1)} Z_{3,2}^{(2)} x_2 Z_{2,1}^{(3)} x_1 (w_0|1^{(p+3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)}). \end{aligned}$$

The first summand on the right is one of our familiar terms. What we have to do is make sense of the remaining two summands. We do this by invoking the Capelli identities (C). These give us:

$$\begin{aligned} &(1) \quad Z_{3,2}^{(2)} Z_{2,1}^{(3)} x_1 = Z_{2,1}^{(1)} x_1 Z_{3,1}^{(2)} +_2 Z_{2,1}^{(2)} x_1 Z_{3,2}^{(1)} Z_{3,1}^{(1)} + Z_{2,1}^{(3)} x_1 Z_{3,2}^{(2)}; \\ &(2) \quad Z_{4,3}^{(1)} Z_{3,2}^{(2)} x_2 = Z_{3,2}^{(1)} x_2 Z_{4,2}^{(1)} + Z_{3,2}^{(2)} x_2 Z_{4,3}^{(1)}. \end{aligned}$$

Applying (1) to the term

$$Z_{4,3}^{(1)} x_3 Z_{3,2}^{(2)} Z_{2,1}^{(3)} x_1 (w_0|1^{(p+3)})(w_1|5^{(1)}) \cdots (w_{q-3}|(q+1)^{(1)}),$$

and observing that the second place doesn't occur, we're left with

$$Z^{(1)}_{4,3}x_3Z^{(1)}_{2,1}x_1(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}).$$

Applying (2) to the term

$$Z^{(1)}_{4,3}Z^{(2)}_{3,2}x_2Z^{(3)}_{2,1}x_1(w_0|1^{(p+3)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}),$$

observing that the second and third places don't occur, and that $Z^{(1)}_{4,3}Z^{(3)}_{2,1}x_1 = Z^{(3)}_{2,1}x_1Z^{(1)}_{4,3}$, we get

$$\begin{aligned}
&Z^{(1)}_{4,3}Z^{(2)}_{3,2}x_2Z^{(3)}_{2,1}x_1(w_0|1^{(p+3)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}) = \\
&Z^{(1)}_{3,2}x_2Z^{(1)}_{4,2}Z^{(3)}_{2,1}x_1(w_0|1^{(p+3)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}) = \\
&Z^{(1)}_{3,2}x_2Z^{(2)}_{2,1}x_1Z^{(1)}_{4,1}(w_0|1^{(p+3)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}) = \\
&Z^{(1)}_{3,2}x_2Z^{(2)}_{2,1}x_1(w_0|1^{(p+2)}4^{(1)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}).
\end{aligned}$$

Thus we see that the boundary of

$$Z^{(1)}_{4,3}x_3Z^{(2)}_{3,2}x_2Z^{(3)}_{2,1}x_1(w_0|1^{(p+3)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)})$$

is

$$\begin{aligned}
&Z^{(1)}_{4,3}x_3Z^{(2)}_{3,2}x_2(w_0|1^{(p)}2^{(3)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}) - \\
&Z^{(1)}_{4,3}x_3Z^{(1)}_{2,1}x_1(w_0|1^{(p+1)}3^{(2)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}) + \\
&Z^{(1)}_{3,2}x_2Z^{(2)}_{2,1}x_1(w_0|1^{(p+2)}4^{(1)})(w_1|5^{(1)})\cdots(w_{q-3}|(q+1)^{(1)}).
\end{aligned}$$

This is precisely the result (up to sign) that we get in the non-letter-place formulation. (Perhaps a word should be added about the signs in these two versions of these resolutions. Given the sequence $k_0, k_1\cdots, k_{q-n}$, satisfying the conditions:

$$\begin{aligned}
k_0 &= 1, \\
k_j - k_{j-1} &= i_{j-1} > 0, \\
\text{and} \qquad q+2-k_{q-n} &> 0,
\end{aligned}$$

we can of course define the sequence $\{i_0, i_1\cdots, i_{q-n}\}$, where $i_{j-1} = k_j - k-$ $i = 1, \cdots, q-n$, and $i_{q-n} = q+2-k_{q-n}$. We then map the term

$$\begin{aligned}
&Z(i_{q-n};k_{q-n})Z(i_{q-n-1};k_{q-n-1})\cdots Z(i_1;k_1)\cdot \\
&Z(i_0;k_0)(w_0|k_0^{(p-1+i_0)})(w_1|k_1^{(i_1)})\cdots(w_{q-n}|k_{q-n}^{(i_{q-n})})
\end{aligned}$$

to the term

$$(-1)^{k_1+\cdots+k_{q-n}}w_0\otimes w_1\otimes\cdots\otimes w_{q-n}\in D_{p-1+i_0}\otimes D_{i_1}\otimes\cdots\otimes D_{i_{q-n}},$$

which provides us with an isomorphism of complexes.)

We now proceed to the definition of the first three steps of a splitting contracting homotopy for this resolution of the hook Weyl module, i.e., of $K_{(p,1^q)}$. The standard basis of this module consists of standard tableaux:

$$\begin{bmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_p} \\ x_{j_1} & & & \\ x_{j_2} & & & \\ \vdots & & & \\ x_{j_q} & & & \end{bmatrix},$$

where the subscripted letters are basis elements of our underlying free module (or, in more orthodox terminology, are words in the letter alphabet), $i_1 \leq i_2 \leq \cdots \leq i_p$ and $i_1 < j_1 < \cdots < j_q$. If we denote by $s_{-1} : K_{(p,1^q)} \to D_p \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q}$ the first step of the homotopy, we define

$$s_{-1}\left(\begin{bmatrix} x_{i_1} & x_{i_2} & \cdots & x_{i_p} \\ x_{j_1} & & & \\ x_{j_2} & & & \\ \vdots & & & \\ x_{j_q} & & & \end{bmatrix}\right) = \left[\begin{array}{l|l} x_{i_1}x_{i_2}\cdots x_{i_p} & 1^{(p)} \\ x_{j_1} & 2 \\ x_{j_2} & 3 \\ \vdots & \vdots \\ x_{j_q} & q+1 \end{array}\right].$$

Double standard tableaux of the type

$$\left[\begin{array}{l|l} w_1 & 1^{(p)} \\ w_2 & 2 \\ w_3 & 3 \\ \vdots & \vdots \\ w_{q+1} & q+1 \end{array}\right] \in D_p \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q},$$

we will call VS (for very standard). On such elements, the next stage of the homotopy

$$s_0 : D_p \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q} \to$$

$$\begin{aligned} &\left\{Z^{(1)}_{2,1}x_1(w_1|1^{(p+1)})(w_3|3)(w_4|4)\cdots(w_{q+1}|q+1)\right\} \oplus \\ &\left\{Z^{(1)}_{3,2}x_2(w_1|1^{(p)})(w_2|2^{(2)})(w_4|4)\cdots(w_{q+1}|q+1)\right\} \oplus \\ &\vdots \qquad\qquad\qquad\qquad \oplus \\ &\left\{Z^{(1)}_{q,q-1}x_{q-1}(w_1|1^{(p)})(w_2|2)\cdots(w_{q-1}|(q-1)^{(2)})(w_{q+1}|q+1)\right\} \oplus \\ &\left\{Z^{(1)}_{q+1,q}x_q(w_1|1^{(p)})(w_2|2)\cdots(w_{q-1}|q-1)(w_q|(q)^{(2)})\right\} \end{aligned}$$

will take the value zero. (The notation is designed to indicate the types of terms we have in dimension 1 of our resolution. For instance, the first term is $D_{p+1} \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q-1}$ prefixed by $Z^{(1)}_{2,1}x_1$, the second is $D_p \otimes D_2 \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q-2}$ prefixed by $Z^{(1)}_{3,2}x_2$, etc. The place notation indicates which place is left out in this tensor product and also which place has degree equal to two [or $p+1$ in the first instance]: in general, the terms prefixed by $Z^{(1)}_{i+1,i}x_i$ have the (i+1)th place omitted and the i^{th} place increased in degree.)

Since we have already defined the value of s_0 to be zero on the VS tableaux of $D_p \otimes \underbrace{D_1 \otimes \cdots \otimes D_1}_{q}$, we have to define s_0 on the double standard tableaux that are not VS. Such a double standard tableau must either be of the form (again we leave out the words and focus on the places):

$$(I) \quad \left[\begin{array}{c|cccc} . & 1^{(p)} & & j & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right] \quad \text{or } (II) \quad \left[\begin{array}{c|cccc} . & 1^{(p)} & & & \\ & 2 & & j & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right].$$

In the case $p = 1$, of course, the second type of tableau would not appear. We define

$$s_0\left(\left[\begin{array}{c|cccc} . & 1^{(p)} & & 2 & \cdots \\ & x & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & y & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right]\right) = Z^{(1)}_{2,1}x_1 \left[\begin{array}{c|cccc} . & 1^{(p+1)} & & & \cdots \\ & x & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & y & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right];$$

$$\begin{aligned}
& s_0\left(\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right]\right) \\
= \quad & Z^{(1)}_{j,j-1} x_{j-1} \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right] - \\
& s_0\left(\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right]\right), j \text{ greater than } 2;
\end{aligned}$$

while

$$s_0\left(\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & & \\ & 2 & & 3 & \cdots \\ & \vdots & & \vdots & \vdots \\ & & & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right]\right) = Z^{(1)}_{3,2} x_1 \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & & \\ & 2^{(2)} & & & \cdots \\ & \vdots & & \vdots & \vdots \\ & & & \cdots & \cdots \\ . & \vdots & \vdots & \vdots & \vdots \end{array}\right];$$

$$
\begin{aligned}
& s_0\left(\left[\begin{array}{c|cccc}
\cdot & 1^{(p)} & & & \\
 & 2 & & j & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & j-1 & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right]\right) \\
= \quad & Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|cccc}
\cdot & 1^{(p)} & & & \\
 & 2 & & j-1 & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & j-1 & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right] - \\
& s_0\left(\left[\begin{array}{c|cccc}
\cdot & 1^{(p)} & & & \\
 & 2 & & j-1 & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & j & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right]\right), j \text{ greater than } 3.
\end{aligned}
$$

With this inductive definition, it is trivial to check that we have the first two steps of a splitting contracting homotopy. It is also clear that a basis for the first syzygies is parametrized by the linearly independent elements

$$
\begin{aligned}
&\left\{ Z^{(1)}_{2,1}x_1\left[\begin{array}{c|cccc}
\cdot & 1^{(p+1)} & & & \cdots \\
 & x & & \cdots & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & y & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right], Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|cccc}
\cdot & 1^{(p)} & & j-1 & \cdots \\
 & 2 & & \cdots & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & j-1 & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right], \right. \\
&\left. Z^{(1)}_{3,2}x_1\left[\begin{array}{c|cccc}
\cdot & 1^{(p)} & & & \\
 & 2^{(2)} & & & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right], Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|cccc}
\cdot & 1^{(p)} & & & \\
 & 2 & & j-1 & \cdots \\
 & \vdots & & \vdots & \vdots \\
 & & j-1 & \cdots & \cdots \\
. & \vdots & \vdots & \vdots & \vdots
\end{array}\right]\right\}.
\end{aligned}
$$

To define the next step of the homotopy, we have to consider the various double standard tableaux that occur in dimension 1 and indicate where they are to be sent. In order to facilitate the description of the map s_1, it's useful to observe that the map s_0, as defined on the tableaux above, could be regarded as a formal operator on those tableaux in the sense that the dots may be filled in in any way we like (even if the terms wouldn't make sense in our particular context). We will see this principle used in what follows; this should clarify what we mean.

It is clear that we may define the map s_1 on all the elements in the brackets above to be zero. Thus all the terms of the form $Z^{(1)}_{2,1}x_1\,[\;|]$ are sent to zero. We now consider elements of the form $Z^{(1)}_{j,j-1}x_{j-1}T$, where $j > 2$ and T is a double standard tableau. There are a number of different cases that have to be considered.

$$1)\quad T \;=\; \left[\begin{array}{c|cccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(\varepsilon)} & \cdots \\ & \vdots & \vdots & & & \vdots \\ \cdot & & & & & \end{array}\right],\ 1 < i < j-1;\ \varepsilon = 1,2$$

$$2)\quad T \;=\; \left[\begin{array}{c|cccccc} \cdot & 1^{(p)} & & & & \\ & 2 & i & \cdots & (j-1)^{(\varepsilon)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right],\ 2 < i < j-1;\ \varepsilon = 1,2$$

$$3)\quad T \;=\; \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & i & \cdots \\ & \vdots & & & \\ & & (j-1)^{(\varepsilon)} & \cdots & \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right],\quad i < j-1;\ \varepsilon = 1,2$$

$$4)\quad T \;=\; \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & & \\ & 2 & i & \cdots & & \\ & \vdots & & & & \\ & & & (j-1)^{(\varepsilon)} & \cdots & \\ \cdot & & & \vdots & & \vdots \end{array}\right],\quad i < j-1;\ \varepsilon = 1,2.$$

In all of these cases, we set $s_1(Z^{(1)}_{j,j-1}x_{j-1}T) = Z^{(1)}_{j,j-1}x_{j-1}s_0(T)$. (Here is where we are treating s_0 as the formal operator alluded to above; we may simply disregard the fact that the map is not legitimately applicable to the tableau , since the formula for $s_0(T)$ used before just depends on the indices less than or equal to i.)

$$5)\quad T = \left[\begin{array}{c|ccc} \cdot & 1^{(p)} & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & & \vdots \end{array}\right].$$

For this situation, we extend the definition of the operator s_0 as follows:

$$s_0\left(\left[\begin{array}{c|ccc} \cdot & 1^{(p)} & 2^{(2)} & \cdots \\ \cdot & \vdots & & \vdots \end{array}\right]\right) = Z^{(2)}_{2,1}x_1\left[\begin{array}{c|cc} \cdot & 1^{(p+2)} & \cdots \\ \cdot & \vdots & \vdots \end{array}\right];$$

$$
s_0 \left[\begin{array}{c|llll}
\cdot & 1^{(p)} & & (j-1)^{(2)} & \cdots \\
 & 2 & & \cdots & \vdots \\
 & \vdots & & & \vdots \\
 & & j-2 & \cdots & \\
\cdot & \vdots & \vdots & & \vdots
\end{array}\right]
$$

$$
= \quad Z^{(2)}_{j-1,j-2} x_{j-2} \left[\begin{array}{c|llll}
\cdot & 1^{(p)} & & (j-2)^{(2)} & \cdots \\
 & 2 & & \cdots & \vdots \\
 & \vdots & & & \vdots \\
 & & j-2 & \cdots & \\
\cdot & \vdots & \vdots & & \vdots
\end{array}\right] -
$$

$$
s_0 \left(\left[\begin{array}{c|lllll}
\cdot & 1^{(p)} & & j-2 & j-1 & \cdots \\
 & 2 & & \cdots & \vdots & \\
 & \vdots & & & \vdots & \\
 & & j-1 & \cdots & & \\
\cdot & \vdots & \vdots & & & \vdots
\end{array}\right] \right).
$$

Here we set

$$
s_1(Z^{(1)}_{j,j-1} x_{j-1} T) \quad = \quad Z^{(1)}_{j,j-1} x_{j-1} s_0(T) +
$$

$$
s_1 \left(Z^{(1)}_{j-1,j-2} x_{j-2} \left[\begin{array}{c|llll}
\cdot & 1^{(p)} & & (j-2)^{(2)} & \cdots \\
 & 2 & & \cdots & \vdots \\
 & \vdots & & & \vdots \\
 & & j & \cdots & \\
\cdot & \vdots & \vdots & & \vdots
\end{array}\right] \right),
$$

where, when $j = 3$, there is no additional summand, and the explicitly exhibited tableau shows that the index j is in the place originally occupied by $j-2$ in the tableau T.

$$
6) \quad T = \left[\begin{array}{c|lll}
\cdot & 1^{(p)} & & \\
 & 2 & (j-1)^{(2)} & \cdots \\
 & \vdots & \vdots & \vdots \\
 & j-2 & \cdots & \\
\cdot & \vdots & & \vdots
\end{array}\right] \quad j > 3.
$$

Again we extend the definition of s_0 :

$$
s_0 \left(\left[\begin{array}{c|lll}
\cdot & 1^{(p)} & & \\
 & 2 & 3^{(2)} & \cdots \\
\cdot & \vdots & \vdots & \vdots
\end{array}\right] \right) = Z^{(2)}_{3,2} x_2 \left[\begin{array}{c|lll}
\cdot & 1^{(p)} & & \\
 & 2^{(3)} & & \cdots \\
\cdot & \vdots & \vdots & \vdots
\end{array}\right];
$$

$$s_0\left(\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & & \\ & 2 & (j-1)^{(2)} & \cdots & \\ & \vdots & \vdots & & \vdots \\ & j-2 & & \cdots & \\ \cdot & \vdots & \vdots & & \vdots \end{array}\right]\right) = Z^{(2)}_{j-1,j-2}x_{j-2}\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & & \\ & 2 & (j-2)^{(2)} & \cdots & \\ & \vdots & \vdots & & \vdots \\ & j-2 & \cdots & \cdots & \\ \cdot & \vdots & \vdots & & \vdots \end{array}\right] -$$

$$s_0\left(\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & & \\ & 2 & (j-2) & j-1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots \\ & j-1 & \cdots & & \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right]\right).$$

With this operation, we can define

$$\begin{aligned} s_1(Z^{(1)}_{j,j-1}x_{j-1}T) &= Z^{(1)}_{j,j-1}x_{j-1}s_0(T) + \\ & s_1\left(Z^{(2)}_{j-1,j-2}x_{j-2}\left[\begin{array}{c|ccc} \cdot & 1^{(p)} & & \\ & 2 & (j-2)^{(2)} & \cdots \\ & \vdots & \vdots & \vdots \\ & j-3 & \cdots & \\ & j & \cdots & \\ \cdot & \vdots & \vdots & \vdots \end{array}\right]\right), \end{aligned}$$

where we recall that $j > 3$, and when $j = 4$ there is no additional summand.

$$7a) \quad T = \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j+k & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right], \quad k \geq 1;$$

$$7b) \quad T = \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j+k & \cdots \\ & \vdots & & \vdots & \\ & & (j-1)^{(2)} & & \\ & & & & \\ \cdot & \vdots & & & \vdots \end{array}\right], \ k \geq 1.$$

In these situations we set

a)

$$
\begin{aligned}
& s_1\left(Z^{(1)}_{j,j-1} x_{j-1} \left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j+1 & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j-1 & & \\
 & & j-1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array} \right] \right) \\
= \; & Z^{(1)}_{j+1,j} x_j Z^{(2)}_{j,j-1} x_{j-1} \left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j-1 & & \\
 & & j-1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array} \right] + \\
& Z^{(1)}_{j+1,j} x_j s_0 \left(\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \vdots \\
 & & & j & \cdots & \\
 & & j & \cdots & & \\
\cdot & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array} \right] \right);
\end{aligned}
$$

$$
\begin{aligned}
& s_1\left(Z^{(1)}_{j,j-1} x_{j-1} \left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j+k & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j-1 & & \\
 & & j-1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array} \right] \right) \\
= \; & Z^{(1)}_{j+k,j+k-1} x_{j+k-1} Z^{(1)}_{j,j-1} x_{j-1} \left[\begin{array}{c|cccccc}
\cdot & 1^{(p)} & & & & j+k-1 & \cdots \\
 & \vdots & & & \vdots & \vdots & \vdots \\
 & & & & j+k-1 & & \\
 & & & j-1 & & & \\
 & & j-1 & & & \vdots & \\
\cdot & \vdots & \vdots & & & & \vdots
\end{array} \right] + \\
& s_1\left(Z^{(1)}_{j,j-1} x_{j-1} \left[\begin{array}{c|cccccc}
\cdot & 1^{(p)} & & & & j+k-1 & \cdots \\
 & \vdots & & & \vdots & \vdots & \vdots \\
 & & & & j+k & & \\
 & & & j-1 & & & \\
 & & j-1 & & & \vdots & \\
\cdot & \vdots & \vdots & & & & \vdots
\end{array} \right] \right);
\end{aligned}
$$

b)

$$
s_1\left(Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j+1 & \cdots \\
 & \vdots & & & \vdots & \vdots \\
 & & & (j-1)^{(2)} & \cdots & \\
 & & & \cdots & & \\
\cdot & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right]\right)
$$

$$
= \quad Z^{(1)}_{j+1,j}x_jZ^{(2)}_{j,j-1}x_{j-1}\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \vdots \\
 & & & (j-1)^{(2)} & \cdots & \\
 & & & \cdots & & \\
\cdot & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right]+
$$

$$
Z^{(1)}_{j+1,j}x_js_0\left(\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \vdots \\
 & & & (j)^{(2)} & \cdots & \\
 & & & \cdots & & \\
\cdot & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right]\right);
$$

$$
s_1\left(Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j+k & \cdots \\
 & \vdots & & & \vdots & \vdots \\
 & & & (j-1)^{(2)} & \cdots & \\
 & & & \cdots & & \\
\cdot & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right]\right)
$$

$$
= \quad Z^{(1)}_{j+k,j+k-1}x_{j+k-1}Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|cccccc}
\cdot & 1^{(p)} & & & & j+k-1 & \cdots \\
 & \vdots & & & \vdots & \vdots & \vdots \\
 & & & & j+k-1 & & \\
 & & & (j-1)^{(2)} & & & \\
 & & & & & \vdots & \\
\cdot & \vdots & \vdots & & & & \vdots
\end{array}\right]
$$

$$
+ \qquad s_1\left(Z^{(1)}_{j,j-1}x_j\left[\begin{array}{c|cccccc}
\cdot & 1^{(p)} & & & & j+k-1 & \cdots \\
 & \vdots & & & \vdots & \vdots & \vdots \\
 & & & & j+k & & \\
 & & & (j-1)^{(2)} & & & \\
 & & & & & \vdots & \\
\cdot & \vdots & \vdots & & & & \vdots
\end{array}\right]\right).
$$

Up to now we have seen the definition of our maps s_{-1}, s_0, and s_1, but have had no demonstration that these do, indeed, define a homotopy. I'll now give some examples of how one shows this so that we can see how trivial it is to verify that these formulae do the job they're supposed to do. It will become clear that the trick lies in finding the formulae. It can only be hoped that, as more proficiency is gained with this letter-place technique, the degree of complication will decrease.

Let's take a double standard tableau of type (I), i.e., we have the tableau:

$$T = \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right],$$

and we are assuming that $j > 2$ (the case $j = 2$ is easy to handle). This goes to zero under the boundary map, so what we must show is that

$$\partial s_0(T) = T.$$

By the definition of s_0, we have

$$s_0(T) = Z^{(1)}_{j,j-1} x_{j-1} \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right] - s_0 \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right].$$

(Don't forget that $j > 2$.) If we apply the boundary map to $s_0(T)$, we get

$$\left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j-1 & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right] + \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right] - \partial s_0 \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right].$$

By a suitable induction hypothesis (starting, of course, with $j = 2$), we may assume that

$$\partial s_0 \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right] = \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & j-1 & \cdots \\ & 2 & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & j & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right]$$

so that we end up with our desired result. (The fact that $\partial s_0(T) = T$ if $T = \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & 2 & \cdots \\ & x & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & y & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right]$,i.e., if $j = 2$, comes from the fact that

$$s_0 \left[\begin{array}{c|cccc} \cdot & 1^{(p)} & & 2 & \cdots \\ & x & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & y & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right] = Z^{(1)}_{2,1} x_1 \left[\begin{array}{c|cccc} \cdot & 1^{(p+1)} & & & \cdots \\ & x & & \cdots & \cdots \\ & \vdots & & \vdots & \vdots \\ & & y & \cdots & \cdots \\ \cdot & \vdots & \vdots & \vdots & \vdots \end{array}\right],$$

so that, with only this one term, we immediately get our result.) The same type of calculation verifies our homotopy for tableaux of type (II).

The next step is a little more complicated; we will demonstrate that for tableaux, T, of type 1) and 7a), the formula

$$s_0 \partial (Z^{(1)}_{j,j-1} x_{j-1} T) + \partial s_1 (Z^{(1)}_{j,j-1} x_{j-1} T) = Z^{(1)}_{j,j-1} x_{j-1} T,$$

is valid. First we consider the tableau of type 1),with $\varepsilon = 2$:

$$T = \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right], \quad 1 < i < j-1.$$

We then have

$$s_0 \partial (Z^{(1)}_{j,j-1} x_{j-1} T) + \partial s_1 (Z^{(1)}_{j,j-1} x_{j-1} T) =$$

$$s_0 \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(1)} j & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right] + \partial (Z^{(1)}_{j,j-1} x_{j-1} s_0(T)) =$$

$$s_0 \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(1)} j & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right] +$$

$$\partial \left(Z^{(1)}_{j,j-1} x_{j-1} s_0 \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right] \right).$$

But

$$\partial\left(Z^{(1)}_{j,j-1}x_{j-1}s_0\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right]\right)=$$

$$Z^{(1)}_{j,j-1}x_{j-1}\partial\left(s_0\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right]\right)-$$

$$Z^{(1)}_{j,j-1}s_0\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right]$$

$$=Z^{(1)}_{j,j-1}x_{j-1}\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(2)} & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right]-s_0\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(1)}j & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right]=$$

$$Z^{(1)}_{j,j-1}x_{j-1}T-s_0\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & i & \cdots & (j-1)^{(1)}j & \cdots \\ \cdot & \vdots & \vdots & & & \vdots \end{array}\right].$$

This gives us our result for tableaux of type 1) with $\varepsilon = 2$. The case $\varepsilon = 1$ proceeds in the same way; the main point is that the operator s_0 is sensitive to the position of the index, i, and doesn't pay any attention to the indices j and $j-1$. The cases 2), 3) and 4) proceed very similarly.

We'll now look at the situation when our tableau, T, is of the form 7a), i.e.,

$$T=\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j+k & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right],\quad k\geq 1.$$

Here, too, we must verify that

$$s_0\partial(Z^{(1)}_{j,j-1}x_{j-1}T)+\partial s_1(Z^{(1)}_{j,j-1}x_{j-1}T)=Z^{(1)}_{j,j-1}x_{j-1}T.$$

As before, we calculate $s_0\partial(Z^{(1)}_{j,j-1}x_{j-1}T)$, and get

$$s_0\left(\begin{array}{c}\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j+k & \cdots \\ & \vdots & & & \vdots & \\ & & & j & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]+ \\ \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j+k & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]\end{array}\right).$$

If $k = 1$, the above equals

$$Z^{(1)}_{j+1,j} x_j \partial_{j,j-1} \left(\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array} \right] \right) -$$

$$s_0 \partial_{j+1,j-1} \left(\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array} \right] \right).$$

(This is seen directly from the definition of the operator s_0.) Working at these terms a little more, we see that the above is

$$
\begin{aligned}
& Z^{(1)}_{j+1,j} x_j \left\{
\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j & & \\
 & & j-1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array}\right] +
\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j-1 & & \\
 & & j & & & \\
\cdot & \vdots & & & & \vdots
\end{array}\right]
\right\} - \\
& Z^{(1)}_{j,j-1} x_{j-1} \left\{
\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j+1 & & \\
 & & j-1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array}\right] +
\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j-1 & & \\
 & & j+1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array}\right]
\right\} \\
& +s_0 \left\{
\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j+1 & & \\
 & & j & & & \\
\cdot & \vdots & & & & \vdots
\end{array}\right] +
\left[\begin{array}{c|ccccc}
\cdot & 1^{(p)} & & & j-1 & \cdots \\
 & \vdots & & & \vdots & \\
 & & & j & & \\
 & & j+1 & & & \\
\cdot & \vdots & & & & \vdots
\end{array}\right]
\right\}.
\end{aligned}
$$

We next have, when $k = 1$,

$$
\begin{aligned}
s_1(Z^{(1)}_{j,j-1}x_{j-1}T) &= Z^{(1)}_{j+1,j}x_jZ^{(2)}_{j,j-1}x_{j-1}\left[\begin{array}{c|cccccc} \cdot & 1^{(p)} & & & & j-1 & \cdots \\ & \vdots & & & & \vdots & \\ & & & & j-1 & & \\ & & & j-1 & & & \\ \cdot & \vdots & & & & & \vdots \end{array}\right] + \\
& \qquad Z^{(1)}_{j+1,j}x_js_0\left(\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j-1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j & & \\ & & j & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]\right),
\end{aligned}
$$

so that $\partial s_1(Z^{(1)}_{j,j-1}x_{j-1}T)$ becomes

$$
\begin{aligned}
& Z^{(1)}_{j+1,j}x_j\left\{\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j & \cdots \\ & \vdots & & & \vdots & \\ & & & j & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right] + \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right] + \right. \\
& \left.\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j-1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j & & \\ & & j & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]\right\} - Z^{(1)}_{j,j-1}x_{j-1}\left\{\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j+1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]\right. \\
& \left. + \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j-1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j+1 & & \\ & & j-1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right] + \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j-1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j-1 & & \\ & & j+1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]\right\} \\
& -s_0\left\{\left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j-1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j+1 & & \\ & & j & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right] + \left[\begin{array}{c|ccccc} \cdot & 1^{(p)} & & & j-1 & \cdots \\ & \vdots & & & \vdots & \\ & & & j & & \\ & & j+1 & & & \\ \cdot & \vdots & & & & \vdots \end{array}\right]\right\}
\end{aligned}
$$

Taking care of signs, one sees that all the terms that should cancel do cancel, and we're left with the one term that we want. Once the case $k = 1$ is settled, the proof for arbitrary k proceeds by induction (the inductive definition of the homotopy is rigged so that this is possible).

4. References

[B-R 1] Projective Resolutions of Weyl Modules. (with G-C Rota). Proc. Natl. Acad. Sci. USA, vol 90, pp 2448–2450, (March, 1993)

[B-R 2]A New Construction in Homological Algebra. (with G-C Rota). Proc. Natl. Acad. Sci. USA, vol 91, pp 4115–4119, (May, 1994)