ON COMMUTATIVE ALGEBRA AND CHARACTERISTIC-FREE REPRESENTATION THEORY

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Abstract. A description of the thread holding together commutative algebra, homological algebra and representation theory. A discussion of some of the developments in the area of determinantal ideals, Schur and Weyl modules, and their interrelationships.

1. Introduction

Over the next five weeks there are going to be two conferences here in Genoa: one on commutative algebra, and one on representation theory (Schur and Weyl modules). More specifically, the titles of these conferences are, *Commutative Algebra, Homological Algebra and Representation Theory*. A paper that best exemplifies the thread that holds these conferences together is the wonderful article by Hilbert [20] on the general theory of algebraic forms. Since I love to talk about that paper, I'll explain very briefly why I say that it provides this common thread.

Hilbert was concerned with the fundamental problems of invariant theory: given a linear group, G, acting on the ring of polynomials, $S = K[X_1, \ldots, X_N]$, we let S^G be the subring of invariants. Is S^G finitely generated as an algebra over the field, K, and if so, what are its generators? Assuming it is finitely generated, i.e., that $S^G = K[Y_1, \ldots, Y_{N'}]/I$, is it the case that I is finitely generated as an ideal in $K[Y_1, \ldots, Y_{N'}]$? (The ideal, I, is called *the ideal of relations on the invariants.*) To answer the latter question, Hilbert proved his famous *Basis Theorem*, undeniably one of the cornerstones of commutative algebra.

Because G acts linearly on the variables of S, it's clear that the ring S^G is graded and that the graded piece of degree ν, $(S^G)_\nu$, is a finite-dimensional vector space over the field, K. A question of prime concern was to determine the dimension of this space for all ν, i.e., to evaluate the function $\chi(\nu) = \dim_K((S^G))_\nu$ for all ν (what we today call the *Hilbert function*), and to determine its growth. What Hilbert saw was that this could be done if one could write down a finite free resolution of S^G over the polynomial ring, $K[Y_1, \ldots, Y_{N'}]$ (this gives an explicit computation of the dimension, and also shows that the function is polynomial). He then proceeded to prove the *Syzygy Theorem,* clearly a fundamental result in homological algebra.

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So, starting with a basic problem in representation theory, Hilbert was led to prove two of the building blocks of commutative and homological algebra—need more be said on this score?

2. Some developments on maximal minors

Now we'll take a look at some of the areas of activity which are connected with the ideas mentioned above. Looking in particular at the action of $GL(V)$ on $S(F \otimes V^* \oplus V \otimes G)$, where F, G, and V are finite-rank free modules over K (an arbitrary commutative ring), $GL(V)$ is the general linear group of V acting in the usual way on $F \otimes V^* \oplus V \otimes G$, and $S(X)$ denotes the symmetric algebra generated by X, we're led to ask for a description of the resolution of the ideal of relations on the invariants of this action. This ideal of relations is of course the ideal, I_{p+1}, generated by the minors of order $p+1$ of the generic $m \times n$ matrix over K, where the ranks of V, F, and G are p, m and n respectively. Now in [15], Eagon and Hochster showed that the depth of this ideal is less than or equal to $(m-p)(n-p)$, for $p = 0, \ldots, \min(m-1, n-1)$. (It was long known that this was true if *depth* were replaced by *height*.) For $p = 0$, we're looking at the ideal generated by the indeterminates, or matrix entries, themselves. The Koszul complex is known to be the minimal resolution of this ideal, and we know the many wonderful properties of the Koszul complex. Among them: it's a rigid complex; it's depth sensitive to the ideal it is resolving. Is it possible to describe the resolution of I_{p+1} as a universal complex which is rigid and is depth sensitive to the ideal generated by the minors of order $p+1$ associated to an arbitrary $m \times n$ matrix, i.e., to a map $\varphi : F \to G$? In the early sixties, Eagon and Northcott [16] found a minimal resolution of I_n (we'll assume from now on that $n \leq m$), i.e., the ideal generated by the maximal minors of the generic matrix. In fact, this complex is defined universally, for any map $\varphi : F \to G$, in terms of the divided powers of G^* (the linear dual of G), and the exterior powers of F, and is depth sensitive to the ideal of maximal minors. To be precise, the term in dimension k of this complex, $k \geq 1$, is $D_{k-1}(G^*) \otimes_R \Lambda^{n+k-1} F$. In dimension 0 it is simply $\Lambda^n G \cong R$ (the ground ring). At about the same time I had defined a family of complexes [5] associated not only to φ but to the maps $\Lambda^l \varphi$ for all $l = 1, \ldots, n$. The cokernels of $\Lambda^l \varphi$ all have the same support as the ideal, $I_n(\varphi)$, of maximal minors of the matrix of φ, and all of these complexes have length $m-n+1$ (as does the Eagon-Northcott complex). This gave, in addition to a resolution of the generic ideal of maximal minors, a resolution of the 'generic cokernel', i.e., a resolution of the cokernel of the generic map. (Notice that, since we're assuming that $m \geq n$, this cokernel is a 'torsion module'—the term put in quotes since we're making no assumption about the nature of the ground ring, other than that it's commutative.) All of these complexes are depth sensitive to the ideal of maximal minors, and rigid (I've never tested the rigidity of the Eagon-Northcott complex, but it is most likely rigid too).

But they are too fat! That is, they are far from minimal—more will be said about that later. In any event, using these complexes, Rim and I [9] defined a notion of multiplicity that only relatively recently has proven to be of some interest in the study of singularities (see [17, 22, 23, 24, 25]).

Another theorem related to this material is the Hilbert-Burch Theorem. In the middle to late sixties, this theorem had been proven (independently) by a number of persons, each time for a quite different reason. Burch, herself, had proven the theorem in order to find a lower bound for the power of the maximal ideal in which an ideal of homological dimension one, generated by n elements, can lie. I had proven it in order to try to carry out the Grothendieck lifting process: clearly free modules could be lifted; could modules of homological dimension one and higher be lifted? Using the information gleaned from the work on the maximal minors of a matrix (namely, the Hilbert-Burch Theorem), I could show that an ideal of homological dimension one could be lifted. This is what sensitized me to the fact that one could get information about modules of finite homological dimension by transferring information from the 'tail' of its resolution to its 'head'.

3. Generic free resolutions and hooks

To elucidate the last sentence of the preceding section, let me remind you that the lifting process of Grothendieck was essentially this: Suppose that the local ring R is equal to $S/(x)$, where x is a regular element of S, and that M is a finitely generated $R-$ module.Is it possible to find an $S-$ module, $\widetilde{M}$ such that i) $\widetilde{M}/x\widetilde{M} \simeq M$ and ii) x is regular on $\widetilde{M}$? If so, then $\widetilde{M}$ is called a *lifting* of M.

Clearly, if M is a free $R-$ module, then it can be lifted. Now suppose that M has a free resolution

$$\mathbb{F} : \cdots \overset{A_{n+1}}{\rightarrow} F_n \overset{A_n}{\rightarrow} \cdots \overset{A_2}{\rightarrow} F_1 \overset{A_1}{\rightarrow} F_0$$

whose maps are presented as matrices, A_k, with entries in R. We set

$$\tilde{\mathbb{F}} : \cdots \overset{\tilde{A}_{n+1}}{\rightarrow} \tilde{F}_n \overset{\tilde{A}_n}{\rightarrow} \cdots \overset{\tilde{A}_2}{\rightarrow} \tilde{F}_1 \overset{\tilde{A}_1}{\rightarrow} \tilde{F}_0,$$

where the $\tilde{F}_k$ are the liftings of the F_k and the $\tilde{A}_k$ are liftings of the matrices A_k to S. If the matrices $\tilde{A}_k$ can be so chosen that $\tilde{A}_k\tilde{A}_{k-1} = 0$ for all k, then the short exact sequence

$$0 \rightarrow \tilde{\mathbb{F}} \overset{x}{\rightarrow} \tilde{\mathbb{F}} \rightarrow \mathbb{F} \rightarrow 0$$

yields immediately that $\tilde{\mathbb{F}}$ is acyclic and that $H_0(\tilde{\mathbb{F}})$ is a lifting of M. Thus, it is clear that any module of homological dimension one may be lifted. What if $\text{hd}_R M = 2$? If we assume that M is cyclic, then we may apply the Hilbert-Burch theorem to see that in that case, too, M can be lifted. Because, if $M = R/I$, and $\text{hd}_R M = 2$, then $\text{hd}_R I = 1$, and I is essentially generated by the maximal minors of the map $\varphi_2 : F_2 \to F_1$. Thus, lifting the matrix A_2 corresponding to φ_2 (and the multiplier if there is one), the matrix $\tilde{A}_1$ can be constructed so that the product $\tilde{A}_2\tilde{A}_1 = 0$.

It was in attempting to generalize this approach to modules of hd ≥ 2 that Eisenbud and I were led to study finite free resolutions and prove the structure theorems that are found in [7]. You may recall that the main thrust of both structure theorems was to transfer information from the tail end of the resolution to the front end. In fact, the proofs proceeded by induction on the distance from the tail of the resolution. Since those proofs depended heavily on two of the complexes of the family described in the section above, and we knew that the complex 'resolving' the ideal of maximal minors could be slimmed down to the Eagon-Northcott complex, we tried to see how to slim down all the other complexes in this family [8]. (Rim and I had already done something of this sort to reconstruct the Eagon-Northcott complex, but our methods were extremely primitive at the time [10].) When we had done this, and shown our results to Towber, he pointed out to us that the modules that had come up were representation modules associated to hooks (in fact, they turned out to be the Weyl modules associated to hooks). I believe it was this small connection of determinantal varieties with representations of the general linear group that Eisenbud pointed out to A. Lascoux in '76-'77 in Paris, and led to the suggestion that Lascoux might apply some of the methods of his thesis-in-progress to the study of determinantal ideals. I'm sure that we're all familiar with Lascoux' use of classical representation-theoretic techniques to describe the terms of the resolution of I_{p+1} when the ground ring contains the rational numbers [26]. This not only created a revolution in the whole subject but also signaled a full circle return to the ideas of Hilbert discussed in the introduction.

(Since all these conferences are taking place here in Italy, I should mention that when I visited A. Andreotti in Pisa in the spring of 1966 and gave a lecture on resolutions of determinantal ideals, he and I discussed for quite a while afterward how to use representation-theoretic techniques to describe the terms of the resolutions. Since the ideal itself was independent of the choice of bases for the modules F and G, all the modules in the resolution of I_{p+1} should have been representations of $GL(F) \times GL(G)$—as, indeed, Lascoux demonstrated so conclusively. But how to do it eluded both Andreotti and me.)

In all events, there was still one severe limitation on Lascoux' approach: the ground ring had to contain the rationals. For those of us interested in arbitrary ground rings, this was clearly not a limitation we could accept without a challenge; some people are not too concerned if we have to exclude rings of positive characteristic from the picture, but surely we should not have to exclude the integers!

4. Characteristic-free approach

The above state of affairs was responsible for my attempt to develop a characteristic-free approach to the representation theory of the general linear group that would enable one to investigate the Lascoux complex in a completely general setting. This was by no means the first attempt to do so: in the early 70's, Carter and Lusztig [13] had worked in positive characteristic p when the ground ring was a field, and in the mid-70's, Towber [27] had worked over an arbitrary commutative ground ring, but the category of representation modules he dealt with was not large enough to allow one to apply the homological methods needed to replace the combinatorics of the classical situation. In particular, one needed kernels and cokernels of certain maps, filtrations in place of direct sum decompositions, and some basic exact sequences. As a result, Akin, Weyman and I [3] introduced and developed multilinear techniques to define a large class of representation modules attached to shape matrices, which included the usual Weyl and Schur (or induced) modules associated with partitions and skew shapes. This enabled us, among other things, to reproduce the terms of Lascoux' resolution, and to see if the purported boundary maps (which were clear in low dimensions) would give us a resolution in general. It took very little work to see that they wouldn't; the homology of the complexes so constructed had non-trivial torsion. However, Akin, Weyman and I did succeed [4] in describing a universal resolution of the generic ideal of submaximal minors which, when tensored over the rationals gave the terms of the Lascoux resolution. But the move from the submaximal to the next case was not automatic and, after many examples which seemed to indicate the important role played by $\mathbf{Z}-$of $GL(F)$, Akin and I decided to undertake a systematic study of $\mathbf{Z}-$forms. To illustrate the notion of $\mathbf{Z}-$form, consider the exact sequence of representation modules:

$$0 \to D_{k+1}F \to D_kF \otimes D_1F \to K_{(k,1)}F \to 0.$$

Over the integers, this element of Ext^1 doesn't split (in fact it generates the cyclic group $\mathrm{Ext}^1_{GL(F)}(K_{(k,1)}F, D_{k+1}F)$). If one maps $D_{k+1}F$ into itself by multiplying by an integer, say μ, then one gets an induced exact sequence whose middle term would not in general be, as a $GL(F)$ representation module over the integers, isomorphic to $D_kF \otimes D_1F$. But tensored with the rationals, it would be isomorphic to it. Thus this module woud be called a $\mathbf{Z}-$form of $D_kF \otimes D_1F$.

A more interesting example, and one used heavily in [A-B-W 2], is provided by the following: We fix a positive integer, l, and consider the complex

$$0 \to D^lF \to D_1F \otimes \Lambda^{l-1}F \to D_2F \otimes \Lambda^{l-2}F \to \cdots ,$$

where the boundary map consists of diagonalizing the exterior power and multiplying it with the divided power. This is a complex of length l (over the rationals it is isomorphic to the usual Koszul complex), which is acyclic for the first $\lceil \frac{l}{2} \rceil$ dimensions. The cycles in these dimensions are thus $\mathbf{Z}-$forms of the corresponding hooks.

The first $\mathbf{Z}-$form described above also arises in the context of resolutions of Weyl modules. Suppose, for example, we consider the Weyl module, K_λ, corresponding to the partition $\lambda = (k-1, 2)$. The Jacobi-Trudi formula says that the Schur function associated to this is the determinant of the matrix

$$\begin{pmatrix} h_{k-1} & h_k \\ h_1 & h_2 \end{pmatrix},$$

where h_t is the complete symmetric polynomial of degree t. In this decade of the twentieth century, we replace h_t by D_t, the divided power module of which h_t is the character, and say that in the Grothendieck ring, the class of K_λ is given by the determinant of

$$\begin{pmatrix} D_{k-1} & D_k \\ D_1 & D_2 \end{pmatrix},$$

i.e., by $D_{k-1} \otimes D_2 - D_k \otimes D_1$. This suggests, as it did to Lascoux, that there should be an exact sequence

$$0 \to D_k \otimes D_1 \to D_{k-1} \otimes D_2 \to K_\lambda \to 0,$$

and this is true in characteristic zero. But it is not true over the integers! In fact, what one does get in general is the exact sequence

$$(\mathbb{E}) \quad 0 \to D_{k+1} \to D_k \otimes D_1 \oplus D_{k+1} \to D_{k-1} \otimes D_2 \to K_\lambda \to 0,$$

where the map of D_{k+1} into $D_k \otimes D_1$ is the customary diagonalization, but where the map from D_{k+1} to D_{k+1} is multiplication by 2. Thus, the kernel of the map $D_{k-1} \otimes D_2 \to K_\lambda$ is precisely one of the $\mathbf{Z}-$forms described in the first example.

We see, then, that the study of $\mathbf{Z}-$forms is tied up with Ext groups and (not surprisingly) with resolutions of Weyl modules. Consequently, Akin and I also started to construct projective resolutions of Weyl modules [1, 2].

5. Homological and combinatorial methods

In [1], we showed that for polynomial representations of GL_n of fixed degree, d, the modules $D_{\alpha_1} \otimes \cdots \otimes D_{\alpha_n}$ with $\alpha_1 + \cdots + \alpha_n = d$ are projective (see also [Gr]), and hence direct sums of such modules are also projective. The terms of the resolutions of Weyl modules that we were looking for were such direct sums (the exact sequence, $(\mathbb{E})$, is an example) and, in [2], we proved the existence of such resolutions. The major tool in that proof was the establishment of the exactness of certain short exact sequences [1], and the use of the mapping cone construction. The method of proof that we used to prove exactness of these sequences involved spectral sequences and Schur complexes; more recent proofs have appeared using letter-place methods (see, e.g., [21]).

For Weyl modules having two rows, we were able to give an explicit description of the resolutions using what we called an 'arithmetic Koszul complex'. (At that time we knew that this arithmetic Koszul complex corresponded to a homogeneous strand of the Bar Complex associated to the divided power algebra on one generator, but we didn't see any way to make use of that observation.). Akin and I made a little further progress in describing explicitly the resolutions of three-rowed Weyl modules associated to skew shapes, but the work was getting more and more complicated without any signs of a simplifying underlying pattern. By the middle of the 80's, we had both turned our attention to other problems.

In the early 90's, Gian-Carlo Rota and I decided to take another look at the problem of resolving Weyl modules. One of the first things we did was to convert to letter-place notation (see [14] and [19]) . It then became clear that the morphisms that Akin and I had been using were place polarization operators (derivations), and that the seemingly complicated identities that had come into play in resolving three-rowed skew shapes were nothing other than Capelli identities in high degree. For example, if we let ∂_{ab} denote the place polarization from place b to place a, then the commutator of ∂_{ab} and ∂_{bc} is ∂_{ac}, i.e.,

$$\partial_{ab}\partial_{bc} - \partial_{bc}\partial_{ab} = \partial_{ac}.$$

If one then applies this to the divided powers of these operators, one gets:

$$\begin{aligned} \partial_{ab}^{(l)}\partial_{bc}^{(k)} &= \sum_{\alpha\geq 0}\partial_{bc}^{(k-\alpha)}\partial_{ab}^{(l-\alpha)}\partial_{ac}^{(\alpha)}; \\ \partial_{bc}^{(k)}\partial_{ab}^{(l)} &= \sum_{\alpha\geq 0}(-1)^{\alpha}\partial_{ab}^{(l-\alpha)}\partial_{bc}^{(k-\alpha)}\partial_{ac}^{(\alpha)}. \end{aligned}$$

Even in the two-row case, it turned out that by using the letter-place approach and reformulating the resolution in [2] in terms of the Bar complex, it was possible to write down a splitting homotopy for the resolutions of two-row skew shapes that enabled us to describe a basis for their syzygies [11]. And for the three-row case, a generalization of the Bar complex that we developed in [12], together with the Capelli identities above, helped to put the fairly *ad hoc* methods that had discouraged Akin and me into conceptual form.

As the situation now stands, we can describe explicitly (inductively) the terms of the resolutions of all shapes in the so-called class of *almost skew shapes* (see [21] or [1] for definition of these shapes; see [6] for a description of the terms). In [6], the description rests heavily on the Bar complex construction, and is obtained by repeated use of the fundamental exact sequences mentioned above, and mapping cones. It is the description of the boundary map that still poses a problem; the actions of some of the 'virtual polarization operators' introduced in the construction of the resolutions are not, as they now stand, transitive. Since the Bar complex relies heavily on the transitivity of algebra actions, a straightforward Bar construction cannot presently work. It is true, however, that all of our actions are transitive up to homotopy, and this homotopy is what has to be built into our construction. Whether this will call for yet another generalization of the Bar construction, or whether we will be able to avoid these non-transitive actions is yet to be determined.

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